Riemann Sums

What is an integral??? The area under a curve

Find $\int_{-3}^{6} f(x)$

$A = \frac{1}{2}(2)(2) \quad 2(2) \quad \frac{1}{2}(2)(2)$

$2 + 4 + 2 + \frac{1}{2} + 2$ $\quad A = 5.5u^2$

Find $\int_{-.5}^{2} f(x)$

not possible bc you can't fit to geometric shapes

Riemann Sums - the process of using rectangles to approximate area

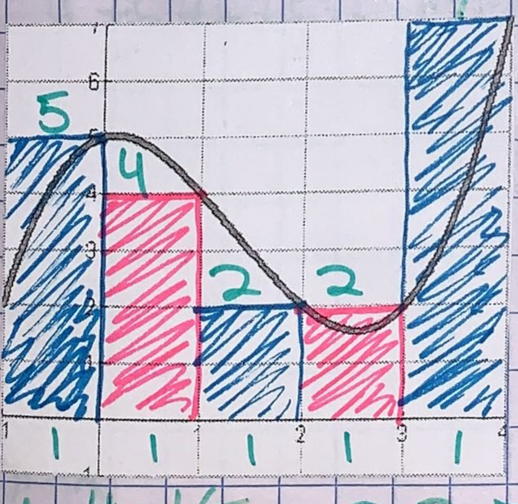
- Right Riemann Sums (RRS)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[f(x_1) + f(x_2) + \ldots f(x_n)\right]$$

- Left Riemann Sums (LRS)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[f(x_0) + f(x_1) + \ldots f(x_n)\right]$$

1. Approximate the integral using the Right Riemann Sums (5 subintervals)

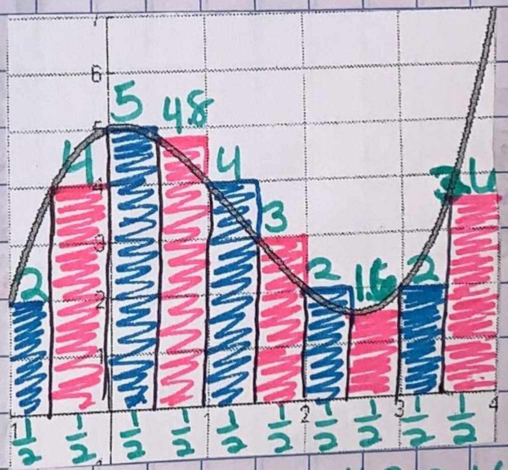


$A = \frac{4 - -1}{5}(5+4+2+2+7)$

$A = 1(5+4+2+2+7)$

$A = 20 \text{ units}^2$

2. Appoximate the integral using the Left Riemann Sums (10 subintervals)



$A = \frac{4 - -1}{10}(2+4+5+4.8+4+3+2+1.6+2+3.6)$

$A = \frac{1}{2}(2+4+5+4.8+4+3+2+1.6+2+3.6)$

$A = 16 \text{ units}^2$

Midpoint Riemann Sums (MRS)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)\dots+f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

3. Approximate the integral using the midpt method (10 subintervals)

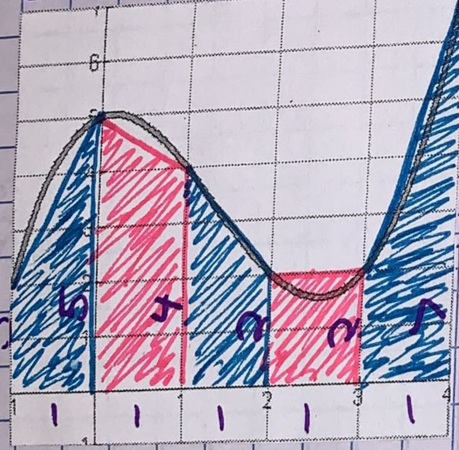
★draw the top of the rectangle through the midpt of the interval

$A=\frac{1}{2}(3+4.6+5+4.5+3.5+2.6+1.8+1.8+2.7+5)$

$A=\frac{1}{2}(34.5)$ $\quad A = 17.25\ u^2$

Trapezoid Rule - It is the avg. of the left + right sums + usually gives a better approximation than either does indiv.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\cdot\frac{b-a}{n}\left[f(x_0)+2f(x_1)+2f(x_{n-1})+f(x_n)\right]$$



★Area of a Trapezoid $=\frac{1}{2}(b_1+b_2)h$

4. Approx. the integral using the trapezoid method (5 subintervals)

$A=\frac{1}{2}(3+5)1+\frac{1}{2}(5+4)1+\frac{1}{2}(4+2)1+\frac{1}{2}(2+2)1+\frac{1}{2}(2+7)$

or

$A=\frac{1}{2}\cdot 1(3+5+5+4+4+2+2+2+2+7)$

$A=18\ u^2$

Riemann Sums - 2 more methods

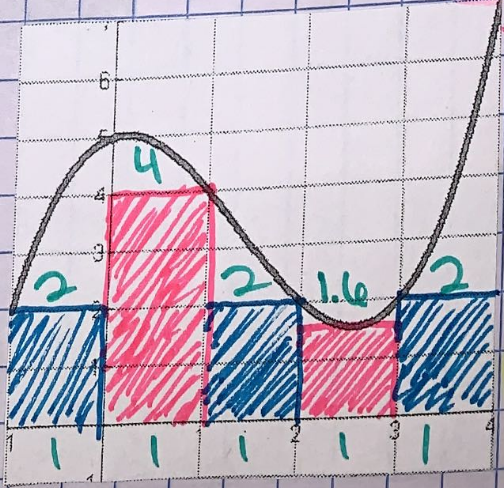
5. Circumscribed (highest point in interval)

★outside curve

$A=1(5+5+4+2+7)$

$A=23\ u^2$

6. Inscribed (lowest pt in interval)



★inside curve

$A=1(2+4+2+1.6+2)$

$A=11.6\ u^2$

The Antiderivative (Indefinite Integrals)

Connection to derivatives:

Find the derivative of...

- $\frac{d}{dx}\, x^2 = 2x$
- $\frac{d}{dx}\, x^2+1 = 2x$
- $\frac{d}{dx}\, x^2+\pi = 2x$
- $\frac{d}{dx}\, x^2+C = 2x$

Antiderivative

What is $2x$ the derivative of?

$x^2,\ x^2+1,\ x^2+\pi \ldots$

$\boxed{x^2+C}$

Antiderivative Theorem:

- The antiderivative of $f(x)$ is the set of functions $F(x)+C$ such that $\frac{d}{dx}[F(x)+C] = f(x)$
- The constant C is the constant of integration.

Notation: $\boxed{\int f(x)\,dx}$ is used to represent the antiderivative of $f(x)$

BASIC INTEGRATION FORMULAS

- $\int k\,dx = kx + C$
- $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$, provided $r \neq 1$
- $\int x^{-1}dx = \int \frac{1}{x}\,dx = \ln|x| + C$
- $\int be^{ax}dx = \frac{b}{a}e^{ax} + C$

Examples:

1. $\int 8\,dx = 8x + C$
2. $\int 3x^2\,dx = \frac{3x^{2+1}}{2+1} = \frac{3x^3}{3} + C = x^3 + C$
3. $\int x^7\,dx = \frac{x^{7+1}}{8} = \frac{x^8}{8} + C$
4. $\int \sqrt{x}\,dx$ ($x^{1/2}$) $= \frac{x^{1/2+1}}{3/2} = \frac{2}{3}x^{3/2} + C$
5. $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-2} = \frac{-1}{2x^2} + C$
6. $\int dx = x + C$ ← understood 1!
7. $\int \sqrt[6]{x^5}\,dx = \int x^{5/6}\,dx = \frac{x^{5/6+1}}{11/6} = \frac{6x^{11/6}}{11} + C$
8. $\int e^x\,dx = e^x + C$
9. $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$
10. $\int \frac{1}{x}\,dx = \ln|x| + C$

$\frac{x^{-1+1}}{0}$ not possible so memorize

6d

MORE ANTIDERIVATIVE RULES

$\int kf(x)dx = k\int f(x)dx$

$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

11. $\int (3x^5 + 7x^2 + 8)\,dx$

$\frac{3x^6}{6} + \frac{7x^3}{3} + 8x + C$

$\frac{x^6}{2} + \frac{7x^3}{3} + 8x + C$

12. $\int \frac{4 + 3x + 2x^4}{x^2}\,dx$

$\int 4x^{-2} + 3x^{-1} + 2x^2\,dx$

$\frac{4x^{-1}}{-1} + 3\ln|x| + \frac{2x^3}{3} + C$

$\frac{-4}{x} + 3\ln|x| + \frac{2}{3}x^3 + C$

13. $\int (2x-3)(x+1)\,dx$

$\int (2x^2 - x - 3)\,dx$

$\frac{2x^3}{3} - \frac{x^2}{2} - 3x + C$

14. $\int \frac{\pi}{x}\,dx = \pi \int \frac{1}{x}\,dx = \pi \ln|x| + C$

TRIG INTEGRALS

$\int \cos x\,dx = \sin x + C$
$\int \sin x\,dx = -\cos x + C$
$\int \sec^2 x\,dx = \tan x + C$
$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$
$\int \csc x \cot x\,dx = -\csc x + C$

15. $\int 2\pi \sin x\,dx$

$2\pi \int \sin x\,dx$

$2\pi(-\cos x) + C$

$-2\pi \cos x + C$

16. $\int \sec^2 x\,dx$

$\tan x + C$

17. $\int \sec x \tan x\,dx = \sec x + C$

18. $\int (5\cos x + 2\sin x)\,dx$

$5\sin x - 2\cos x + C$

19. $\int 3\csc^2 x\,dx$

$3\int \csc^2 x\,dx = 3(-\cot x) + C = -3\cot x + C$

OTHER COMMON INTEGRAL

$\int \frac{1}{x}\,dx$ or $\int x^{-1}dx = \ln\|x\| + c$
$\int e^x dx = e^x + c$
$\int a^x dx = \frac{a^x}{\ln a} + c$

20. $\int 3^x\,dx$

$\frac{3^x}{\ln 3} + C$

Fundamental Theorem of Calculus (Pt. 2)
Definite Integrals

Review: Riemann Sums to Approximate Area

Find the area under the graph $y=x^2+1$ over $[-1,2]$

midpt R.S.

$A \approx 1(1.5+1.5+3.3) \approx 6.3u^2$

The Fundamental Theorem of Calculus Pt 2: Definite Integral Def.

If $F(x)$ is the antiderivative of the continuous function $f(x)$ over the interval $[a,b]$, then the definite integral from a to b is...

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Steps for evaluating definite integrals:

1. Find any ~~antiderivative~~ antiderivative, $F(x)$, of $f(x)$. Top - Bottom
2. Evaluate $F(x)$ using b and a & compute $F(b) - F(a)$. The result is the area under the graph over the interval $[a,b]$.

Now evaluate the definite integral

$$\int_1^2 (x^2+1)\,dx$$

★ This is read "the integral of x^2+1, with respect to x, from -1 to 2."

$$\frac{x^3}{3}+x \Big|_{-1}^{2}$$

$$\left(\frac{2^3}{3}+2\right) - \left(\frac{(-1)^3}{3}+-1\right)$$

Top - Bottom

$$\frac{8}{3}+2+\frac{1}{3}+1 = 6$$

★ Compare to Riemann Sum rev. answer

Remember to distribute the -1 to both terms!

FTC continued...

1. $\int_{-1}^{4} (x^2 - x)\,dx$ $\quad \frac{x^3}{3} - \frac{x^2}{2}\Big|_{-1}^{4} = \left(\frac{4^3}{3} - \frac{4^2}{2}\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)$

$\frac{64}{3} - \frac{16}{2} + \frac{1}{3} + \frac{1}{2} = \boxed{\frac{85}{6}}$

2. $\int_{0}^{3} e^x\,dx$ $\quad e^x\Big|_0^3 = e^3 - e^0 = \boxed{e^3 - 1}$

3. $\int_{1}^{e} \left(1 + 2x - \frac{1}{x}\right) dx$ assume $x > 0$

$x + \frac{\cancel{2}x^2}{\cancel{2}} - \ln|x|\Big|_1^e = (e + e^2 - \ln e) - (1 + 1^2 - \ln 1)$

$e + e^2 - 1 - 1 - 1 + 0 = \boxed{e + e^2 - 3}$

4. $\int_{1}^{4} \left(2\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ $\quad \int_1^4 (2x^{1/2} + x^{-1/2})\,dx$

$\frac{2 \cdot 2x^{3/2}}{3} + \frac{2x^{1/2}}{1} = \frac{4x^{3/2}}{3} + 2x^{1/2}\Big|_1^4$

$\left(\frac{4\sqrt{4}^3}{3} + 2\sqrt{4}\right) - \left(\frac{4\sqrt{1}^3}{3} + 2\sqrt{1}\right) = \frac{32}{3} + 4 - \frac{4}{3} - 2 = \boxed{\frac{34}{3}}$

5. $\int_{0}^{\ln 4} 2e^x\,dx$

use properties of logs to rewrite

$2e^x\Big|_0^{\ln 4} = 2e^{\ln 4} - 2e^0 = \cancel{\ln e}\, 4 \cdot 2 - 2(1) = 8 - 2 = \boxed{6}$

6. $\int_{1}^{5} \frac{x-1}{x}\,dx = \int_1^5 \frac{x}{x} - \frac{1}{x}\,dx = \int_1^5 \left(1 - \frac{1}{x}\right) dx$

← Rewrite

$x - \ln|x|\Big|_1^5 = (5 - \ln 5) - (1 - \ln 1) = 5 - \ln 5 - 1 + 0 =$

$\boxed{4 - \ln 5}$

7. $\int_{\pi}^{5} \cos x\,dx$

$\sin x\Big|_\pi^5 = \sin 5 - \sin \pi = \sin 5 - 0 = \boxed{\sin 5}$

8. $\int_{5}^{-2} 3x^2\,dx$ If the bigger # is not on top, switch & negate.

$-\int_{-2}^{5} 3x^2\,dx = -x^3\Big|_{-2}^{5} = -(5)^3 - -(-2)^3$

$-125 + -8 = \boxed{-133}$

Properties of Integrals

$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$$

Given $\int_{-2}^{7} f(x)\,dx = 25$, $\int_{-2}^{7} g(x)\,dx = 4$, and $\int_{-2}^{4} g(x)\,dx = 1$.
evaluate the following:

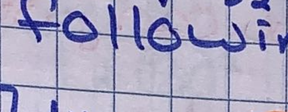
1. $\int_{-2}^{7} [f(x) + g(x)]\,dx = 25 + 4 = 29$

2. $\int_{-2}^{7} [3f(x) - 4g(x)]\,dx = 3(25) - 4(4) = 59$

3. $\int_{7}^{-2} [6g(x)]\,dx = -1\int_{-2}^{7} [6g(x)]\,dx = -6(4) = -24$

   move bigger # to top + multiply by -1

4. $\int_{4}^{7} g(x)\,dx$



   $4 - 1 = 3$

5. $\int_{-2}^{-2} f(x)\,dx = 0$

Fundamental Theorem of Calculus (Part 1)

Part 1: If f is continuous on an open interval containing a, then for every x in the interval,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

The Long Way:

1. $\frac{d}{dx}\int_1^x t^2\,dt$

$$\frac{d}{dx}\left.\frac{t^3}{3}\right|_1^x = \frac{d}{dx}\left(\frac{x^3}{3}-\frac{1}{3}\right)$$

$$\frac{\cancel{3}x^2}{\cancel{3}} - 0 = x^2$$

2. $\frac{d}{dx}\int_3^x \sin t\,dt$

$$\frac{d}{dx}\left(\left.-\cos t\right|_3^x\right) = \frac{d}{dt}(-\cos x - -\cos 3)$$

$$-(-\sin x) + 0 = \sin x$$

Short Cut: When x is the upper limit & a constant is your lower limit, you can just plug in x for t.

Examples with Short cut:

1. $\frac{d}{dx}\int_{-3}^x \sqrt{t^2+1}\,dt$

$$\sqrt{x^2+1}$$

2. $\frac{d}{dx}\int_2^x \csc^2 t\,dt$

$$\csc^2 x$$

3. $\frac{d}{dx}\int_x^2 \sin t^2\,dt = \frac{d}{dx}-1\int_2^x \sin t^2\,dt = -\sin x^2$

more sophisticated use of FTC...

1. $\frac{d}{dx}\int_{\pi/2}^{x^3} \cos t\,dt$

$$\frac{d}{dx}\left.\sin t\right|_{\pi/2}^{x^3} = \frac{d}{dx}\left(\sin x^3 - \sin \pi/2\right) = \frac{d}{dx}\sin(x^3) - \frac{d}{dx}\left(\sin\frac{\pi}{2}\right)$$

$$\cos(x^3)\cdot 3x^2 - 0$$

$$3x^2\cos(x^3)$$

Short Cut: When something other than just x is the upper & a constant is still the lower limit...

1. plug upper into the function
2. take derivative of the upper

Examples with short cut:

1. $\frac{d}{dx}\int_3^{x^2} \sqrt{t^2-4\sin t}\,dt$

$\sqrt{(x^2)^2-4\sin x^2}\cdot 2x$

$2x\sqrt{x^4-4\sin x^2}$

2. $\frac{d}{dx}\int_{x^3}^{3} \frac{1}{t^2}\,dt$

$\frac{d}{dx}\cdot -\int_3^{x^3} \frac{1}{t^2}\,dt$

$\frac{-1}{(x^3)^2}\cdot 3x^2 = \frac{-3}{x^4}$

you can't ignore the lower limit when it isn't a constant!

3. $\frac{d}{dx}\int_{3x}^{4x^2} \frac{4t}{1+t^2}\,dt$

$\frac{4(4x^2)}{1+(4x^2)^2}\cdot 8x - \frac{4(3x)}{1+(3x)^2}\cdot 3 = \frac{128x^3}{1+16x^4} - \frac{36x}{1+9x^2}$