Unit 4B – Applications of Derivatives

- Notes and some practice are included
- Homework will be assigned on a daily basis

Topics Covered:

- ❖ Mean Value Theorem
- ❖ Particle Motion
- ❖ Optimization
- ❖ Implicit Differentiation
- ❖ Related Rates

Quiz is ________________

Test is ________________

Name: ______________

Mean Value Theorem and Rolle's Theorem

Determine whether the Mean Value Theorem can be applied to the function on the indicated interval. If it can be applied, find all values of c that satisfy the theorem.

1. $f(x) = x^2 - 4x$ on the interval $0 \le x \le 4$	2. $f(x) = (x+4)^2(x-3)$ on the interval $-4 \le x \le 3$
3. $f(x) = 4 - \lvert x-2 \rvert$ on the interval $-3 \le x \le 7$	4. $f(x) = \sin x$ on the interval $0 \le x \le 2\pi$
5. $f(x) = x^3 - x^2 - 2x$ on $-1 \le x \le 1$	6. $f(x) = \dfrac{x+2}{x}$ on $\frac{1}{2} \le x \le 2$

Particle Motion

Answer the following questions for each position function $s(t)$ in meters where t is in seconds if a particle is moving along the x-axis.

$s(t) = t^3 - 3t + 3 \quad [0,6]$	$s(t) = t^3 - 6t^2 \quad [0,7]$
a. What is the velocity function?	a. What is the velocity function?
b. What is the velocity at $t = 3$ seconds?	b. What is the velocity at $t = 3$ seconds?
c. When is the particle at rest?	c. When is the particle at rest?
d. When is the particle moving right? Moving left?	d. When is the particle moving right? Moving left?
e. What is the acceleration function?	e. What is the acceleration function?
f. What is the acceleration at $t = 1$ second?	f. What is the acceleration at $t = 1$ second?
g. What is the displacement?	g. What is the displacement?
h. What is the total distance traveled?	h. What is the total distance traveled?
i. When is the particle speeding up? Slowing Down?	i. When is the particle speeding up? Slowing Down?
j. Find the velocity when the acceleration is 0.	j. Find the velocity when the acceleration is 0.

$s(t) = 2t^3 - 21t^2 + 60t + 3 \quad [0,8]$	$s(t) = 2t^3 - 14t^2 + 22t - 5 \quad [0,6]$
a. What is the velocity function?	a. What is the velocity function?
b. What is the velocity at $t = 3$ seconds?	b. What is the velocity at $t = 3$ seconds?
c. When is the particle at rest?	c. When is the particle at rest?
d. When is the particle moving right? Moving left?	d. When is the particle moving right? Moving left?
e. What is the acceleration function?	e. What is the acceleration function?
f. What is the acceleration at $t = 1$ second?	f. What is the acceleration at $t = 1$ second?
g. What is the displacement?	g. What is the displacement?
h. What is the total distance traveled?	h. What is the total distance traveled?
i. When is the particle speeding up? Slowing Down?	i. When is the particle speeding up? Slowing Down?
j. Find the velocity when the acceleration is 0.	j. Find the velocity when the acceleration is 0.

Optimization

1. From a thin piece of cardboard 8" x 8", square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

2. Your friend asks you to produce a box from cardboard that is 30" x 30" with dimensions that will maximize volume. What dimensions will yield a box of maximum volume? What is the maximum volume?

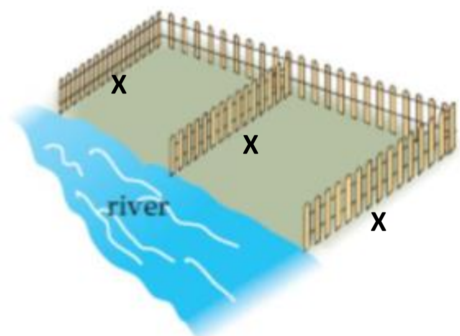
3. A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy duty fencing selling for $3 a foot, while the remaining two sides will use standard fencing selling for $2 a foot. What are the dimensions of the rectangular plot of greatest area that can be fenced for a cost of $6,000?

4. Some airlines place restrictions on the size of luggage that passengers are allowed to take with them. Fly-By-Night Airlines has a rule that the sum of the length, width, and height of any piece of baggage must be less than 158 cm. A passenger wants to take a suitcase that holds the largest volume possible. If the length and width are to be equal, what should be the dimensions of the suitcase, and what would be the maximum volume?

5. A container firm is designing an open-top rectangular box, with a square base, that will hold 108 cubic centimeters (cc). What dimensions will yield the minimum surface area? What is the minimum surface area?

6. A hobby store has 20 feet of fencing to enclose a rectangular area for an electric train in one corner of its display room. The two sides against the wall require no fence. What dimensions of the rectangle will maximize the area? What is the maximum area?

7. A rancher wants to enclose two rectangular areas near a river, one for sheep and one for cattle. There is 250 yards of fencing available. What is the largest total area that can be enclosed?

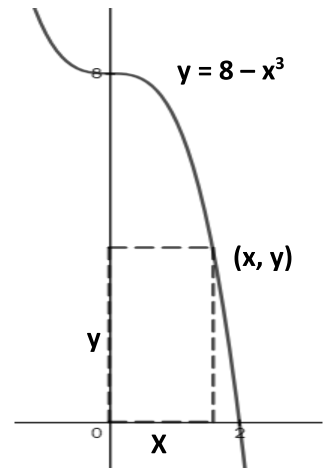


8. A carpenter is building a rectangular room with a fixed perimeter of 54 ft. What are the dimensions of the largest room that can be built? What is its area?

9. What is the largest area that a rectangle can have inscribed in a closed region bounded by the x-axis, y-axis, and the line $y = -3x + 6$?

10. Find the rectangle of maximum area which is inscribed in the closed region bounded by $x = 0$, $y = 0$ and the line $y = -\frac{1}{2}x + 2$.

11. Find the dimensions of the rectangle of largest area, which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.



12. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 6 - 2x^2$. What is the maximum area the rectangle can have and what are its dimensions?

13. A rectangular field is to have 60,000 m^2. Fencing is required to enclose the field and to divide it in half (2 equal areas). What are the outer dimensions of the field that require the <u>minimum</u> amount of fencing?

14. Raggs, Ltd., a clothing firm, determines that in order to sell x suits, the price per suit must be $p = 150 - 0.5x$. It also determines that the total cost of producing x suits is given by $C(x) = 4000 + 0.25x^2$.

a) Find the total revenue.
b) Find the total profit.
c) How many suits must the company produce and sell in order to maximize profit?
d) What is the maximum profit?
e) What price per suit must be charged in order to make this maximum profit?

15. An appliance firm is marketing a new refrigerator. It determines that in order to sell x refrigerators, the price per refrigerator must be $p = 280 - 0.4x$. It also determines that the total cost of producing x refrigerators is given by $C(x) = 5000 + 0.6x^2$.

a) Find the total revenue.
b) Find the total profit.
c) How many refrigerators must the company produce and sell in order to maximize profit?
d) What is the maximum profit?
e) What price per refrigerators must be charged in order to make this maximum profit?

16. Fight promoters ride a thin line between profit and loss, especially in determining the price to charge for admissions to closed-circuit television showings in local theaters. By keeping records, a theater determines that if the admission price is $20, it averages 1000 people in attendance. But for an increase of $1, it loses 100 customers from the average number. Every customer spends an average of $1.80 on concessions. What admission price should the theater change in order to maximize total revenue? How many people will attend at that price?

17. Suppose 1000 hotdogs can be sold every week if sold for $1 each. For every 20 cent increase in price, sales decrease by 100 per week. Find the price p for which each hotdog should be sold in order to maximize the revenue received per week from their sale.

Implicit Differentiation

For each problem, use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

1. $x = = 5y^2 + 1$

2. $3 = 4x^2 + 3\,y^3$

3. $x^2 + 3y^3 = -y^2 + 4$

4. $x^3 = -3y^3 - 4y^2 + 5$

5. $-2x^3 + 3x^3y = -3y^2 + 2$

6. $-2xy + 1 = 2x^3 + 2x^2y^2$

7. $x^2 + y^2 = \sqrt{7}$

8. $\sqrt{x} = 5\sqrt{y}$

9. $\sin(y^2) + x = 7$

10. $\tan(xy) + 5 = 0$

11. $xy = y \sin x$

12. $\cos y = x$

Find the equation of the tangent line to the curve at the given point.

13. $xy^2 = 1$ at $(1, -1)$

14. $y^2 = x^2 y$ at $(-1, 2)$

Related Rates – Cubes, Circles, Spheres, and Squares

1. All edges of a cube are expanding at a rate of 3 cm/sec. How fast is the volume changing when each edge is 1 cm?

2. The volume of a cube is decreasing at a rate of 12 cubic meters per hour. How fast is the total surface area decreasing when the surface area is 24 m^2?

3. The radius of a circle is increasing at the rate of 5 in/min. At what rate is the area increasing when the radius is 10 inches?

4. A stone in a still pond creates a circular ripple whose radius increases at a constant rate of 3 ft/s. At what rate is the area enclosed by the ripple increasing 8 s after the stone strikes the pond?

5. A pebble is dropped into a calm pond creating ripples whose radius increases at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area of the disturbed water changing?

6. The radius of a sphere is increasing at a constant rate of 0.05 cm/sec. At the time when the radius of the sphere is 10 cm, what is the rate of increase of the volume?

7. A spherical balloon is inflated at the rate of four cubic feet per minute. At what rate is the radius changing when $r = 24\ in$?

8. Air is being pumped into a spherical balloon at the rate of 4.5 cubic inches per minute. Find the rate of change of the radius when the radius is 2 inches.

9. How fast is the area of a square increasing when the side is 3 m in length and growing at a rate of 0.8 m/min?

10. A rectangle has a fixed area of 100 $unit^2$. Its length is increasing at 2 units/sec. Find the length at the instant the width is decreasing at 0.5 units/sec.

11. A screen saver displays the outline of a 3 cm by 2 cm rectangle and then expands the rectangle in such a way that the 2 cm side is expanding at the rate of 4 cm/sec and the proportions of the rectangle never change. How fast is the area of the rectangle increasing when its dimensions are 12 cm by 8 cm?

Related Rates – Ladders, Cars, Boats, etc.

1. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the lop of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

2. A ladder leans against a wall with the bottom of the ladder 8 feet from the wall. The top of the ladder slips down the wall at a rate of 4 ft/sec while the bottom of the ladder is being pulled away at a rate of 3 ft/sec. How long is the ladder?

3. If one leg of a right triangle increases at a rate of 2 in/sec, while the other leg decreases at 3 in/sec, find how fast the hypotenuse is changing when the first leg is 6 ft and the other leg is 8 ft.

4. A ladder 15 m tall slides down the side of a water tower. When the bottom end is 11 m from the tower, the opposite end is sliding down at a rate of 3 m/h.
   a. At that instant, how fast is the bottom of the ladder moving away from the tower?

   b. How fast is the area of the region created between the ladder, the ground, and the tower changing?

5. Darth Vader's spaceship is approaching the origin along the positive y axis at 50 km/sec. Meanwhile, his daughter Ella's spaceship is moving away from the origin along the positive x-axis at 80 km/sec. When Darth is at $y = 1200\ km$ and Ella is at $x = 500\ km$, is the distance between them increasing or decreasing? At what rate?

6. A winch at the end of the dock is 9 ft above the level of the deck of a boat. A rope attached to the deck is being hauled in by the winch at a rate of 3 ft/sec. How fast is the boat being pulled toward the dock when 15 ft of rope are out?

7. A boat is pulled toward a pier by means of a taut cable. If the boat is 20 ft below the level of the pier and the cable is pulled in at a rate of 36 ft/min, how fast is the boat moving when it is 48 ft from the base of the pier?

8. Two vehicles are approaching an intersection, one truck from the west at 15 m/sec and one van from the north at 20 m/sec. How fast is the distance between the vehicles changing at the instant the truck is 60 m west and the van 80 m north of the intersection?

9. Car A is going west at 50 mph and car B is headed north at 60 mph. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

10. An angler has hooked a fish. The fish was swimming in an east-west direction along a line 40 ft north of the angler. If the line is leaving the reel at a rate of 7 ft/sec when the fish is 60 ft from the angler, how fast is the fish traveling?

Related Rates Using Similar Triangles – Shadows & Cones

1. A streetlight is 15 feet above the sidewalk. A man 6 feet tall walks away from the light at the rate of 5 ft/sec.
    a. Determine the rate at which the man's shadow is lengthening at the moment that he is 20 feet from the base of the light.

    b. Find the rate at which the tip of the shadow is changing at this time.

2. A man 2 m tall walks away from a lamppost whose light is 5 m above the ground. If he walks at a speed of 1.5 m/s, at what rate is his shadow growing when he is 10 m from the lamppost?

3. Sulley the squirrel, a stunning 1.5 ft tall, is walking away from a 15 ft lamppost at a rate of 6 ft/min and heading home after collecting nuts for the winter. How fast is the length of Sulley's shadow increasing? At what rate is the tip of his shadow changing?

4. A man 6 ft tall walks toward a wall. A light, 30 ft from the wall, is on the ground directly behind the man. If the man is walking at a rate of 4 ft/sec, how fast is the tip of the shadow moving up the wall when he is 5 feet from the wall?

5. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight towards the building at a speed of 1.6 m/sec, how fast is his shadow on the building decreasing when he is 4 meters from the building?

6. A water tank has the shape of an inverted circular cone with base radius 2 m and a height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

7. Water is flowing into an inverted cone at the rate of 5 cubic inches per second. If the cone has an altitude of 4 in and a base radius of 3 in, how fast is the water level rising when the water is 2 in deep? How fast is the radius of the water changing when the water is 2 in deep?

8. At a sand and gravel plant, sand is falling off a conveyer and into a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high?

Unit 4B Derivative Applications Review

1. $s(t) = t^3 - 12t^2$ $[0,10]$
   - k. What is the velocity function?
   - l. What is the velocity at $t = 3$ seconds?
   - m. When is the particle at rest?
   - n. When is the particle moving right? left?
   - o. What is the acceleration function?
   - p. What is the acceleration at $t = 1$ second?
   - q. What is the displacement?
   - r. What is the total distance traveled?
   - s. When is the particle speeding up? Slowing Down?
   - t. Find the velocity when the acceleration is 0.

2. $s(t) = -t^3 + 13t^2 - 40t$ $[0,6]$
   - k. What is the velocity function?
   - l. What is the velocity at $t = 3$ seconds?
   - m. When is the particle at rest?
   - n. When is the particle moving right? left?
   - o. What is the acceleration function?
   - p. What is the acceleration at $t = 1$ second?
   - q. What is the displacement?
   - r. What is the total distance traveled?
   - s. When is the particle speeding up? Slowing Down?
   - t. Find the velocity when the acceleration is 0.

Implicit Differentiation: Find the derivative with respect to x.

3. $x^2 + y^2 = 100$

4. $xy = x$

5. $\frac{1}{x} + \frac{1}{y} = 5$

6. $x^3y + y = 1$

7. $\sin(y^2) = 3x$

8. $\cos(xy) = 2$

9. $x^2 + y + xy^3 = 5$

Find the equation of the tangent line to the curve given.

10. $x^2y + y = 2$ at $(1, 3)$

11. $x^2 + y^2 = 9$ at $(3, 4)$

Optimization & Related Rates:

12. Four pens will be built side by side along a wall by using 150 feet of fencing. What dimensions will maximize the area of the pens.

13. A box with an open top is to be constructed from a square piece of cardboard, 3 feet wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

14. A travel agency is planning tours for groups of 20 or more. If the group size is exactly 20, the cost is $400 per person. For each additional person, the price will be reduced by $15 for every person in the group. What size group will provide the largest revenue?

15. The area of a healing wound is given by $A = \pi r^2$. The radius is decreasing at the rate of 1 mm/day. How fast is the area decreasing at the moment when r = 25 mm?

16. If the radius of a sphere is increasing at the constant rate of 3 mm/sec. How fast is the volume changing when the surface area ($4\pi r^2$) is 10 mm^2?

17. A 13 meter ladder slides down the side of a water tower. At the moment, the bottom end is 12 m from the water tower, the opposite end of the ladder is sliding down at a rate of 3 m/hr. How fast is the bottom of the ladder moving away from the tower?

18. A streetlight is 18 feet above the sidewalk. A man 6 ft. tall walks away from the light at the rate of 15 ft/sec. Determine the rate at which the man's shadow is lengthening at the moment he is 12 ft from the base of the light.

19. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

20. All edges of a cube are expanding at a rate of 6 inches per second. Find the rate of change of the volume of the cube when the area of one side is equal to 25 in^2.

21. Suppose you are drinking root beer from a conical paper cup. The cup has a diameter of 6 cm and a depth of 10 cm. As you drink root beer through a straw, the root beer is leaving the cup at a rate of 7 cm^3/sec. At what rate is the level of the root beer in the cup changing when the root beer is 6 cm deep?

Unit 4B Derivative Applications: Additional Review

Use implicit differentiation to find $\frac{dy}{dx}$.

1. $x^2 + y^2 = 16$
2. $x^3 - xy + y^2 = x$
3. $\sin x + 2\cos(2y) = 1$

Find the equation of the tangent line to the graph of the functions below at the given point.

4. $xy = 4$ @ $(-4, -1)$
5. $x^3 + y^3 = 2xy$ @ $(1,1)$

Optimization

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

7. A real estate office handles 50 apartment units. When the rent is $720 per month, all units are occupied. However, on the average, for each $40 increase in rent, one unit becomes vacant. Each occupied unit requires an average of $48 per month for services and repairs. What rent should be charged to obtain the maximum profit?

8. An open rectangular box (no lid) with a square base has volume of 5000 cubic inches. What dimensions yield the minimum surface area? What is the minimum surface area?

Related Rates

9. A spherical balloon is inflated with gas at the rate of 20 cubic feet per minute. How fast is the radius of the balloon increasing at the instant the radius is 2 feet?

10. At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the height. At what rate is the height of the pile changing when the pile is 15 feet high?

11. A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second. How fast is the top moving down the wall when the base of the ladder is 7 feet from the wall?

12. A man 7 feet tall walking at a rate of 7 ft/sec away from a light that is 14 feet above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing?

Particle Motion

Position is measured in meters and time is in seconds for a particle moving along the x-axis.

Given: $s(t) = 2t^3 - 27t^2 + 108t + 5$ [0,7] sec. Find the following:

13. Velocity as a function of time t.

14. Velocity at t = 3 sec.

15. Acceleration as a function of time t.

16. When is the particle standing still?

17. When is the particle moving right? Moving left?

18. What is the displacement?

19. What is the total distance traveled?

20. When is the particle speeding up? slowing down?

21. Find the velocity when acceleration is 0.