AP Calculus AB – The Meaning of the Derivative

Fall 2020 - Unit 3

Date	Topic	Assignment
Tuesday, September 1st	**Keeper 3.1 The Average Rate of Change & the Definition of a Derivative at a Point** *EQ: What is a derivative?*	Rates of Change & the Derivative (Packet p. 1 - 3)
Wednesday, September 2nd	**Keeper 3.2 The Definition of the Derivative** *What is the derivative of a function at a point and how is it related to the tangent line?*	The Derivative as a Function; Differentiability (Packet p. 4 – 6)
Thursday, September 3rd	**Keeper 3.3 Interpreting the Derivative** *How can you interpret derivatives in the real world?*	**Skills Check 3.1 & 3.2** Interpreting the Derivative (Packet p. 7 - 8) Review – Using the Definition of the Derivative (Packet p. 9 - 10)
Friday, September 4th	**Keeper 3.4 Curve Sketching f to f'** *What does the first derivative and the second derivative tell us about the function?*	**Virtual Quiz 3.1-3.3** Curve Sketching – Graphing f' from f (Packet p. 11 - 12)
Tuesday, September 8th	**Keeper 3.5 Curve Sketching f from f'** *What does the first derivative and the second derivative tell us about the function?*	**Skills Check 3.4** Curve Sketching (Packet p. 13 – 14) Curve Sketching – Graphing f from f' (Packet p. 15 – 16)
Wednesday, September 9th	**Review**	Practice Test – The Meaning of Derivatives (Packet p. 17 – 21)
Thursday, September 10th	**Unit 3 Test**	Good Luck!

Rates of Change and The Derivative

Find an equation for the tangent line and the normal line to the graph of each function at the indicated value.

1. $f(x) = x^2 + 2, x = -1$ $\quad f(-1) = 3$

$\lim_{x\to -1} \frac{x^2+2-3}{x+1}$

$\lim_{x\to -1} \frac{(x+1)(x-1)}{x+1}$

$\lim_{x\to -1} x-1$

$= -2$

Tan Line: $y-3 = -2(x+1)$

Norm Line: $y-3 = \frac{1}{2}(x+1)$

2. $f(x) = x^3 + 1, x = 1$ $\quad f(1) = 2$

$\lim_{x\to 1} \frac{x^3+1-2}{x-1}$

$\lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1}$

$\lim_{x\to 1} x^2+x+1$

$= 1+1+1 = 3$

Tan Line: $y-2 = 3(x-1)$

Norm Line: $y-2 = -\frac{1}{3}(x-1)$

3. $f(x) = \frac{2-5x}{1+x}$ at 0 $\quad f(0) = 2$

$\lim_{x\to 0} \frac{\frac{2-5x}{1+x} - 2}{x-0}$

$\lim_{x\to 0} \frac{2-5x-2-2x}{1+x} \cdot \frac{1}{x}$

$\lim_{x\to 0} \frac{-7}{1+x}$

$= -\frac{7}{1} = -7$

Tan Line: $y-2 = -7x$

Norm Line: $y-2 = \frac{1}{7}x$

4. $f(x) = \sqrt{x+3}, x = 6$ $\quad f(6) = 3$

$\lim_{x\to 6} \frac{\sqrt{x+3}-3}{x-6}$

$\lim_{x\to 6} \frac{x+3-9}{(x-6)(\sqrt{x+3}+3)}$

$\lim_{x\to 6} \frac{1}{\sqrt{x+3}+3}$

$= \frac{1}{3+3} = \frac{1}{6}$

Tan line: $y-3 = \frac{1}{6}(x-6)$

Norm Line: $y-3 = -6(x-6)$

5. $f(x) = \frac{1}{\sqrt{x}}, x = 4$ $\quad f(4) = \frac{1}{2}$

$\lim_{x\to 4} \frac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x-4}$

$\lim_{x\to 4} \frac{2-\sqrt{x}}{2\sqrt{x}} \cdot \frac{1}{x-4}$

$\lim_{x\to 4} \frac{-(\sqrt{x}-2)}{2\sqrt{x}(\sqrt{x}-2)(\sqrt{x}+2)}$

$\lim_{x\to 4} \frac{-1}{2\sqrt{x}(\sqrt{x}+2)}$

$= \frac{-1}{2\cdot 2\cdot 4} = -\frac{1}{16}$

tan Line: $y-\frac{1}{2} = -\frac{1}{16}(x-4)$

Norm Line $y-\frac{1}{2} = 16(x-4)$

6. $f(x) = \frac{1}{x^2}, x = 2$ $\quad f(2) = \frac{1}{4}$

$\lim_{x\to 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{x-2}$

$\lim_{x\to 2} \frac{4-x^2}{4x^2} \cdot \frac{1}{x-2}$

$\lim_{x\to 2} \frac{-(x+2)}{4x^2}$

$= \frac{-4}{16} = -\frac{1}{4}$

Tan Line: $y-\frac{1}{4} = -\frac{1}{4}(x-2)$

Norm Line: $y - 1/4 = 4(x-2)$

Find the rate of change of f at the indicated number.

7. $f(x) = 5x - 2, c = 0$ $\quad f(0) = -2$

$$\lim_{x\to 0} \frac{5x-2-2}{x}$$

$$\lim_{x\to 0} 5$$

$$= 5$$

$$f'(0) = 5$$

8. $f(x) = x^2 - 1, c = -1$ $\quad f(-1) = 0$

$$\lim_{x\to -1} \frac{x^2-1}{x+1}$$

$$\lim_{x\to -1} x-1$$

$$= -2$$

$$f'(-1) = -2$$

9. $f(x) = \frac{x^2}{x+3}, c = 0$ $\quad f(0) = 0$

$$\lim_{x\to 0} \frac{\frac{x^2}{x+3}}{x}$$

$$\lim_{x\to 0} \frac{x}{x+3} = 0$$

$$f'(0) = 0$$

10. $f(x) = \frac{x}{x^2-1}, c = 2$ $\quad f(2) = 2/3$

$$\lim_{x\to 2} \frac{\frac{x}{x^2-1} - \frac{2}{3}}{x-2}$$

$$\lim_{x\to 2} \frac{3x-2x^2+2}{3(x^2-1)} \cdot \frac{1}{x-2}$$

$$\begin{array}{r|rrr} 2 & -2 & 3 & 2 \\ & & -4 & -2 \\ \hline & -2 & -1 & \end{array}$$

$$\lim_{x\to 2} \frac{-2x-1}{3(x^2-1)}$$

$$= \frac{-5}{9}$$

$$f'(2) = -5/9$$

Find the derivative of each function at the given number.

11. $f(x) = 2x + 3$ at 1 $\quad f(1) = 5$

$$\lim_{x\to 1} \frac{2x+3-5}{x-1}$$

$$\lim_{x\to 1} \frac{2(x-1)}{x-1} = 2$$

$$f'(1) = 2$$

12. $f(x) = 3x^2 + x + 5$ at -1 $\quad f(-1) = 7$

$$\lim_{x\to -1} \frac{3x^2+x+5-7}{x+1}$$

$$\begin{array}{r|rrr} -1 & 3 & 1 & -2 \\ & & -3 & 2 \\ \hline & 3 & -2 & \end{array}$$

$$\lim_{x\to -1} 3x-2$$

$$f'(-1) = -5$$

13. The line $x + y = 5$ is tangent to the graph of $y = f(x)$ at the point where $x = 2$. What are the values of $f(2)$ and $f'(2)$?

$y = -x + 5$ tan Line

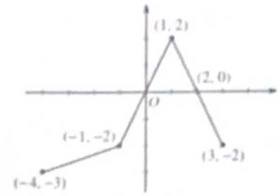
$m = -1 = f'(2)$

$f(2) = -2 + 5 = 3$

$\therefore \quad f'(2) = -1$

$f(2) = 3$

14. The graph of the function f, given below, consists of three line segments. Find $f'(0)$.



Graph of f

$f'(0) = \frac{2-(-2)}{1-(-1)} = \frac{4}{2} = 2$

15. What is the instantaneous rate of change of the function $f(x) = 3x^2 + 5$ at $x = 2$? $f(2) = 17$

$\lim_{x\to 2} \frac{3x^2+5-17}{x-2}$

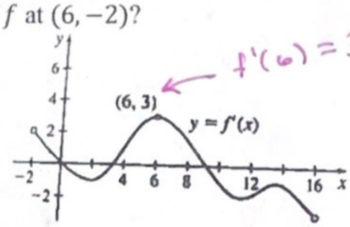
$\lim_{x\to 2} \frac{3(x^2-4)}{x-2}$

$\lim_{x\to 2} \frac{3(x+2)(x-2)}{x-2}$

$\lim_{x\to 2} 3(x+2) = 12$

$f'(2) = 12$

16. The function f is defined on the closed interval $[-2,16]$. The graph of the derivative of f, $y = f'(x)$, is given below. The point $(6, -2)$ is on the graph of $y = f(x)$. What is the equation of the graph of f at $(6, -2)$?



$f'(6) = 3$

pt : $(6, -2)$
$\downarrow \quad \downarrow$
$x_1 \quad y_1$

$y + 2 = 3(x - 6)$

17. If $x - 3y = 13$ is an equation of the normal line to the graph of f at the point $(2,6)$, then what is the value of $f'(2)$?

$x - 3y = 13$

$3y = x - 13$

$y = \frac{1}{3}x - \frac{13}{3}$ Norm Line at $x=2$

$f'(2) = -3$

18. A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 hours the water is continuously used and no water is added to replace it. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

Use the table to approximate $A'(5)$.

$\frac{66-71}{5-2} = -5/3$

$\frac{60-66}{7-5} = -6/2 = -3$

$\frac{-5/3 + (-3)}{2} = \frac{-\frac{14}{3}}{2} = \frac{-7}{3}$

$A'(5) \approx -7/3$

The Derivative as a Function; Differentiability

Find the derivative of each function at any real number c.

1. $f(x) = 10$

$\lim_{h\to 0} \frac{10-10}{h} = 0$

$f'(x) = 0$

2. $f(x) = 2x + 3$

$\lim_{h\to 0} \frac{2(x+h)+3-2x-3}{h}$

$\lim_{h\to 0} \frac{2h}{h} = 2$

$f'(x) = 2$

3. $f(x) = 2 - x^2$

$\lim_{h\to 0} \frac{2-(x+h)^2-2+x^2}{h}$

$\lim_{h\to 0} \frac{2-x^2-2xh-h^2-2+x^2}{h}$

$\lim_{h\to 0} -2x-h = -2x$

$f'(x) = -2x$

Differentiate each function f and determine the domain of f'.

4. $f(x) = 5$

$\lim_{h\to 0} \frac{5-5}{h} = 0$

$f'(x) = 0$

Domain f': $\mathbb{R}$

5. $f(x) = 3x^2 + x + 5$

$\lim_{h\to 0} \frac{3(x+h)^2+x+h+5-3x^2-x-5}{h}$

$\lim_{h\to 0} \frac{3x^2+6xh+3h^2+x+h+5-3x^2-x-5}{h}$

$\lim_{h\to 0} \frac{6xh+3h^2+h}{h}$

$\lim_{h\to 0} 6x+3h = 6x$

$f'(x) = 6x$

Domain f': $\mathbb{R}$

6. $f(x) = 5\sqrt{x-1}$

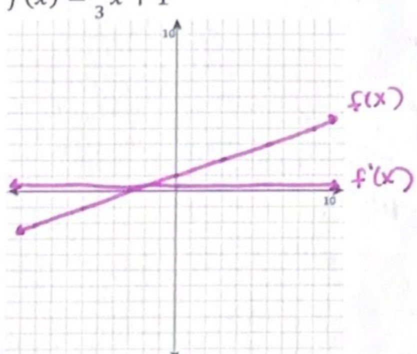
$\lim_{h\to 0} \frac{5\sqrt{x+h-1}-5\sqrt{x-1}}{h}$

$\lim_{h\to 0} \frac{25(x+h-1)-25(x-1)}{h(5\sqrt{x+h-1}+5\sqrt{x-1})}$

$\lim_{h\to 0} \frac{25}{5\sqrt{x+h-1}+5\sqrt{x-1}} = \frac{5}{2\sqrt{x-1}}$ Domain f': $(1,\infty)$

Differentiate each function f. Graph $y = f(x)$ and $y = f'(x)$ on the same set of coordinate axes.

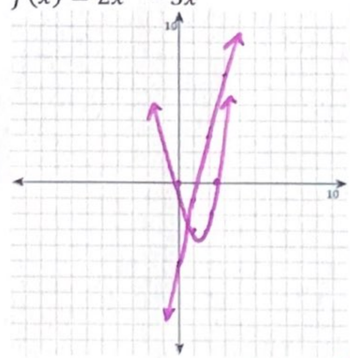
7. $f(x) = \frac{1}{3}x + 1$



$\lim_{h\to 0} \frac{\frac{1}{3}(x+h)+1-\frac{1}{3}x-1}{h}$

$\lim_{h\to 0} \frac{\frac{1}{3}h}{h} = \frac{1}{3}$

8. $f(x) = 2x^2 - 5x$ $f(x) = x(2x-5)$



$\lim_{h\to 0} \frac{2(x+h)^2-5(x+h)-2x^2+5x}{h}$

$\lim_{h\to 0} \frac{2x^2+4xh+2h^2-5x-5h-2x^2+5x}{h}$

$\lim_{h\to 0} 4x+2h-5 = 4x-5$

Find the derivative of each function.

9. $f(x) = mx + b$

$$\lim_{h\to 0} \frac{m(x+h)+b-mx-b}{h}$$

$$\lim_{h\to 0} \frac{mh}{h}$$

$$f'(x) = m$$

10. $f(x) = ax^2 + bx + c$

$$\lim_{h\to 0} \frac{a(x+h)^2+b(x+h)+c-ax^2-bx-c}{h}$$

$$\lim_{h\to 0} \frac{ax^2+2axh+ah^2+bx+bh+c-ax^2-bx-c}{h}$$

$$\lim_{h\to 0} \frac{2axh+ah^2+bh}{h}$$

$$\lim_{h\to 0} 2ax+ah+b = 2ax+b$$

$$f'(x) = 2ax+b$$

11. $f(x) = \frac{1}{x^2}$

$$\lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$\lim_{h\to 0} \frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2}$$

$$\lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

$$f'(x) = -\frac{2}{x^3}$$

12. $f(x) = \frac{1}{\sqrt{x}}$

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$$

$$\lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

$$\lim_{h\to 0} \frac{x-x-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

$$\lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})}$$

$$= \frac{-1}{x\cdot 2\sqrt{x}}$$

$$= \frac{-1}{2x^{3/2}}$$

$$f'(x) = \frac{-1}{2x^{3/2}}$$

Each limit represents the derivative of a function f at some point c. Determine f and c.

13. $\lim\limits_{h\to 0} \frac{(2+h)^2-4}{h}$

$f(x) = x^2$

$a = 2$

14. $\lim\limits_{x\to 1} \frac{(x^2-1)}{x-1}$

$f(x) = x^2$

$a = 1$

16. $\lim\limits_{h\to 0} \frac{\sqrt{9+h}-3}{h}$

$f(x) = \sqrt{x}$

$a = 9$

17. $\lim\limits_{x\to \frac{\pi}{6}} \frac{\sin x - \frac{1}{2}}{x-\frac{\pi}{6}}$

$f(x) = \sin x$

$a = \pi/6$

18. $\lim\limits_{x\to 0} \frac{2(x+2)^2-(x+2)-6}{x}$

$f(x) = 2(x+2)^2-(x+2)$

$a = 0$

19. $\lim\limits_{h\to 0} \frac{(3+h)^2+2(3+h)-15}{h}$

$f(x) = x^2+2x$

$a = 3$

For problems 20-24, use the function $f(x) = \frac{x}{x+2}$

20. Find $f'(x)$ by finding $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0} \frac{\frac{x+h}{x+h+2} - \frac{x}{x+2}}{h}$$

$$\lim_{h\to 0} \frac{x^2+2x+hx+2h-x^2-xh-2x}{h(x+h+2)(x+2)}$$

$$\lim_{h\to 0} \frac{2}{(x+h+2)(x+2)}$$

$$= \frac{2}{(x+2)^2}$$

$$f'(x) = \frac{2}{(x+2)^2}$$

21. Find the slope of the normal line drawn to the graph of $f(x)$ at $x = -2$

$$m_{Tan}\big|_{x=-2} = \frac{2}{(-2+2)^2} \quad \text{DNE}$$

$$\therefore\ m_{Norm} \ \text{DNE}$$

22. Find the slope of the tangent line drawn to the graph of $f(x)$ at $x = -1$

$$m_{Tan}\big|_{x=-1} = \frac{2}{(-1+2)^2} = 2$$

23. Find the equation of the tangent line drawn to the graph of $f(x)$ at $x = 6$

$$m_{tan}\big|_{x=6} = \frac{2}{(6+2)^2} = \frac{2}{64} = \frac{1}{32}$$

$$f(6) = \frac{6}{6+2} = \frac{6}{8} = \frac{3}{4}$$

$$y - 3/4 = \frac{1}{32}(x-6)$$

24. Find $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, where $a = 1$ $\quad f(1) = 1/3$

$$\lim_{x\to 1} \frac{\frac{x}{x+2} - \frac{1}{3}}{x-1}$$

$$\lim_{x\to 1} \frac{3x-x-2}{(x-1)(3(x+2))}$$

$$\lim_{x\to 1} \frac{2}{3(x+2)} = \frac{2}{9}$$

$$f'(1) = 2/9$$

Interpreting the Derivative

1. Suppose the cost of drilling x feet for an oil well is $C = f(x)$ dollars
   a. What are the units of $f'(x)$? $\frac{dc}{dx}$ = \$/foot
   b. In practical terms, what does $f'(x)$ mean in this case?
      The IROC in cost when x feet of oil is drilled
   c. What can you say about the sign of $f'(x)$?
      $f'(x) > 0$ since you would expect the cost to increase as you drill deeper.
   d. Estimate the cost of drilling an additional foot, starting at the depth of 300 ft, given that $f'(300) = 1000$
      At a depth of 300 ft the cost of drilling one additional foot will cost \$1000.

2. A paint manufacturing company estimates that it can sell $g = f(p)$ gallons of paint at a price of p dollars.
   a. What are the units of $\frac{dg}{dp}$
      $\frac{dg}{dp}$ = gallons/dollar
   b. In practical terms, what does $\frac{dg}{dp}$ mean in this case?
      The IROC in gallons of paint when the cost is p dollars.
   c. What can you say about the sign of $\frac{dg}{dp}$?
      $\frac{dg}{dp} < 0$ since as price increases, the # of gallons sold will decrease
   d. Given that $\frac{dg}{dp} = -100$ when $p = 20$, what can you say about the effect of increasing the price from \$20 per gallon to \$21 per gallon?
      Increasing the price from \$20 to \$21 will decrease the number of gallons by 100 gallons.

3. If $q = f(p)$ gives the number of cords of wood sold when the price per cord is p dollars.
   a. What are the units of $f'(p)$? $\frac{dq}{dp}$ = cords of wood/dollars
   b. In practical terms, what does $f'(p)$ mean in this case?
      The IROC in cords of wood when the cost is p dollars.
   c. What can you say about the sign of $f'(p)$?
      $\frac{dq}{dp} < 0$ since as price increases the # of cords sold will decrease.
   d. Given that $f'(425) = -\frac{2}{5}$, what can you say about the effect of decreasing the price of a cord of wood from \$425 to \$420?
      If the price is decreased from \$425 to \$420, you would expect to sell 2 additional cords.

4. If $q = f(p)$ gives the number of pounds of sugar produced when the price per pound is p dollars, then what are the units and the meaning of the statement $f'(3) = 50$?

The IROC in pounds of sugar when the price per pound is \$3 is 50 pounds per price per pound

The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars. What is the meaning of the derivative $f'(x)$? What are the units? What does the statement $f'(800) = 17$ mean?

a) $f'(x)$ means the IROC in cost when producing x ounces of gold

b) $f'(x)$ = dollars/ounce

c) The IROC in cost when 800 ounces of gold is mined is 17 dollars/ounce

5. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$. What is the meaning of $f'(5)$? What are its units?

a) The IROC in the # of bacteria after 5 hours

b) $f'(t)$ = bacteria/hr

6. Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion?

$f'(10)$ would be bigger b/c the bacteria would be growing at a faster rate

If the supply of nutrients is limited, $f'(5)$ would likely be bigger b/c the bacteria would be dying instead of growing

7. Let T(t) be the temperature (in °F) in Phoenix t hours after midnight on September 10, 2008. The table shows the values of this function recorded every two hours. What is the meaning of T ' (8)? Estimate its value.

t	0	2	4	6	8	10	12	14
T	82	75	74	75	84	90	93	94

The IROC in temp 8 hrs after midnight is 3.75° F per hour

$M_{sec(6-8)} = \frac{84-75}{2} = \frac{9}{2} = 4.5$

$M_{sec(8-10)} = \frac{90-84}{2} = \frac{6}{2}\ 3$

$M_{Avg} = \frac{4.5+3}{2} = \frac{7.5}{2} = \frac{\frac{15}{2}}{2} = \frac{15}{4} = 3.75$

Review - Using the Definition of the Derivative

Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

1. $\lim_{h\to 0} \frac{(1+h)^{10}-1}{h}$

$f(x) = x^{10}$

$a = 1$

2. $\lim_{h\to 0} \frac{\sqrt[4]{16+h}-2}{h}$

$f(x) = \sqrt[4]{x}$

$a = 16$

3. $\lim_{x\to 5} \frac{2^x-32}{x-5}$

$f(x) = 2^x$

$a = 5$

4. $\lim_{x\to \pi/4} \frac{\tan x-1}{x-\pi/4}$

$f(x) = \tan x$

$a = \pi/4$

5. $\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h}$

$f(x) = \cos x$

$a = \pi$

6. $\lim_{t\to 1} \frac{t^4+t-2}{t-1}$

$f(t) = t^4 + t$

$a = 1$

Find the derivative of the function using the definition using the definition of derivative. State the domain of the function and the domain of its derivative.

7. $f(x) = ax^2 + b$

$\lim_{h\to 0} \frac{a(x+h)^2+b-ax^2-b}{h}$

$\lim_{h\to 0} \frac{ax^2+2axh+ah^2+b-ax^2-b}{h}$

$\lim_{h\to 0} 2ax + ah$

$= 2ax$

$f'(x) = 2ax$

$D_{f(x)} = \mathbb{R}$

$D_{f'(x)} = \mathbb{R}$

8. $f(x) = x^2 - 2x^3$

$\lim_{h\to 0} \frac{(x+h)^2 - 2(x+h)^3 - x^2 + 2x^3}{h}$

$\lim_{h\to 0} \frac{x^2+2xh+h^2-2(x^3+3x^2h+3xh^2+h^3)-x^2+2x^3}{h}$

$\lim_{h\to 0} \frac{2xh+h^2-6x^2h-6xh^2-2h^3}{h}$

$\lim_{h\to 0} 2x+h-6x^2-6xh-2h^2$

$= 2x-6x^2$

$D_{f(x)} = \mathbb{R}$

$D_{f'(x)} = \mathbb{R}$

9. $G(t) = \frac{1-2t}{3+t}$

$$\lim_{h\to 0} \frac{\frac{1-2t-2h}{3+t+h} - \frac{1-2t}{3+t}}{h}$$

$$\lim_{h\to 0} \frac{3+t-6t-2t+t-6h-2ht-3-t-h+6t+2t^2+2th}{h(t+3)(3+t+h)}$$

$$\lim_{h\to 0} \frac{-6h-h}{h(3+t)(3+t+h)}$$

$$\lim_{h\to 0} \frac{-7}{(3+t)(3+t+h)} = \frac{-7}{(3+t)^2}$$

$D_{f(t)} = (-\infty, -3) \cup (-3, \infty)$

$D_{f'(t)} = (-\infty, -3) \cup (-3, \infty)$

10. $f(x) = x^{\frac{3}{2}}$

$$\lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h}$$

$$\lim_{h\to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h}$$

$$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h(\sqrt{(x+h)^3} + \sqrt{x^3})}$$

$$\lim_{h\to 0} \frac{3x^2h + 3xh^2}{h(\sqrt{(x+h)^3} + \sqrt{x^3})}$$

$$\lim_{h\to 0} \frac{3x^2 + 3xh}{\sqrt{(x+h)^3} + \sqrt{x^3}}$$

$$= \frac{3x^2}{2x^{3/2}} = 3/2\sqrt{x}$$

$D_{f(x)} = [0, \infty)$

$D_{f'(x)} = [0, \infty)$

11. $g(t) = \frac{1}{\sqrt{t}}$

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h}$$

$$\lim_{h\to 0} \frac{\sqrt{t} - \sqrt{t+h}}{h\sqrt{t}\sqrt{t+h}}$$

$$\lim_{h\to 0} \frac{t - t - h}{h\sqrt{t}\sqrt{t+h}(\sqrt{t} + \sqrt{t+h})}$$

$$\lim_{h\to 0} \frac{-1}{\sqrt{t}\sqrt{t+h}(\sqrt{t} + \sqrt{t+h})}$$

$$= \frac{-1}{t \cdot 2\sqrt{t}}$$

$$= \frac{-1}{2t^{3/2}}$$

$D_{g(t)} = (0, \infty)$

$D_{g'(t)} = (0, \infty$

12. $f(x) = x^4$

$$\lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$$

$$\lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$\lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 = 4x^3$$

$D_{f(x)} = \mathbb{R}$

$D_{f'(x)} = \mathbb{R}$

13. Let $T(t)$ be the average monthly temperature (measured in degrees Fahrenheit) in Atlanta where t is measured in months since January 2000. What is the meaning of each of the following?

a.) $T(4) = 72$ In May of 2000 the average temp is 72°F

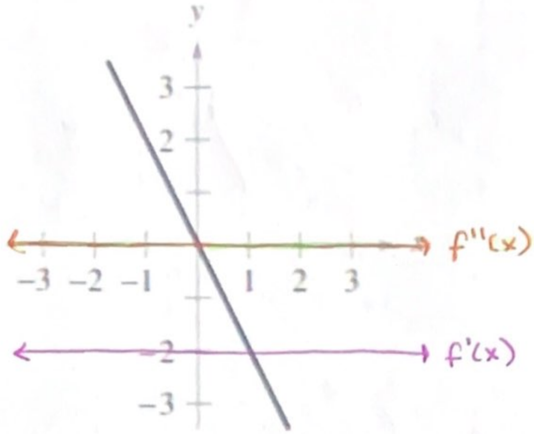
b) $T'(11) = -8$ The IROC in temp in Dec of 2000 is -8°F per Month

c) $T'(14) = 12$ The IROC in temp in March of 2001 is 12°F per month

Curve Sketching - Graphing f' from f

The graph of f is given below. Sketch a possible graph of f' and f''

1.



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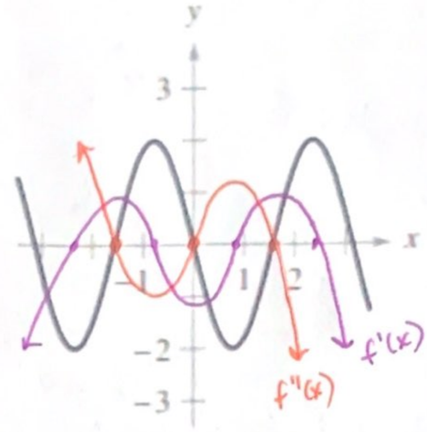

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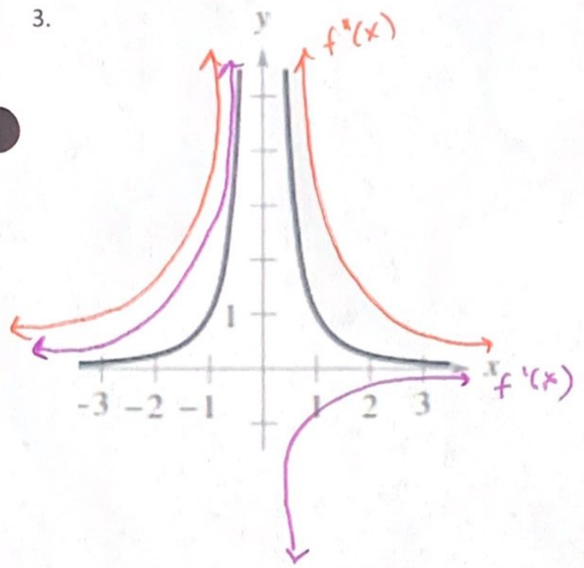

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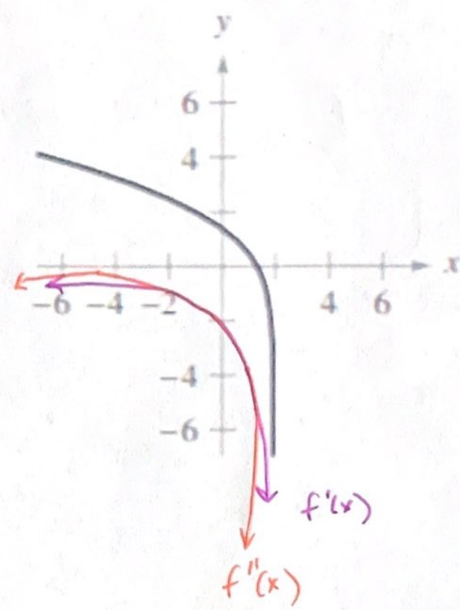

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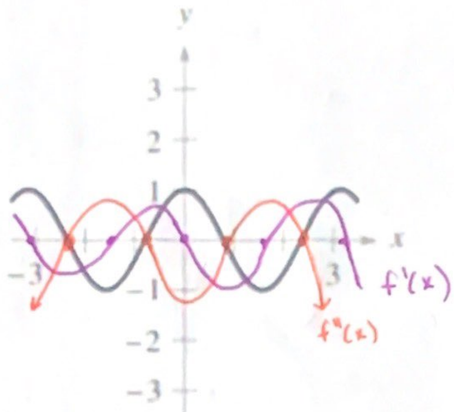
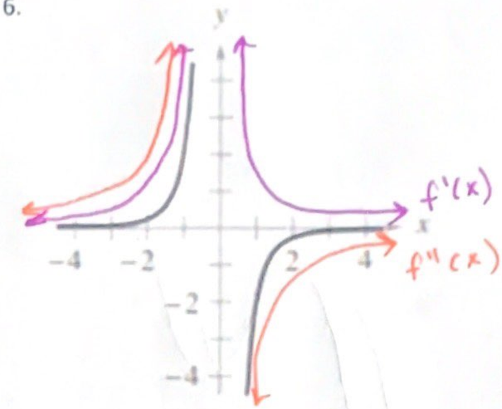


6.



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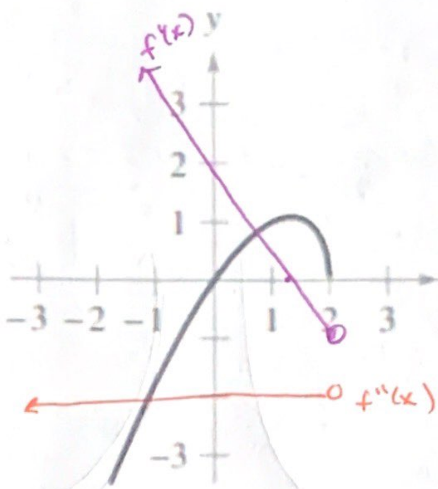

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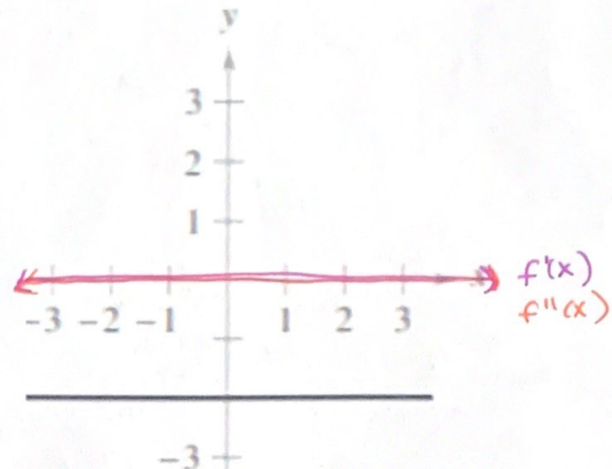
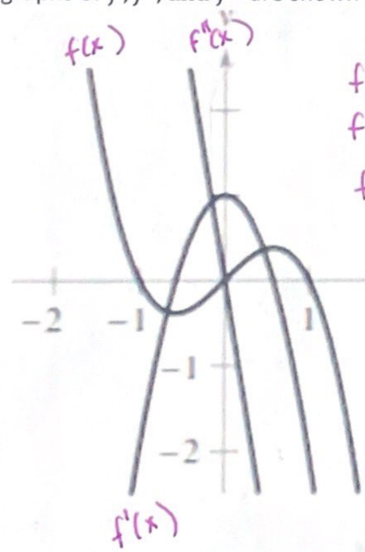


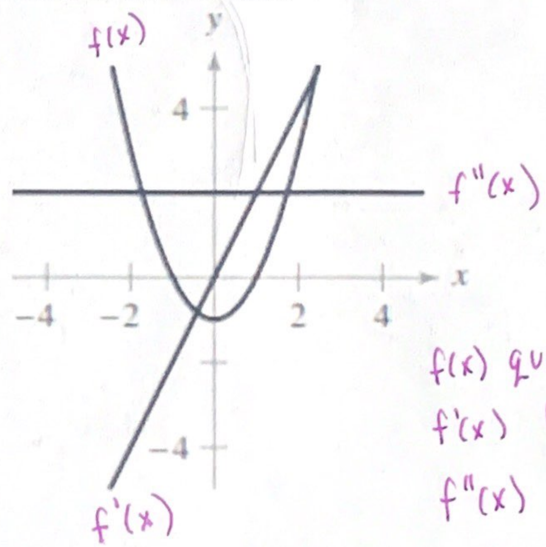
The graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which?

9.



f(x) cubic
f'(x) quadratic
f''(x) Linear

10.



f(x) quadratic
f'(x) linear
f''(x) horizontal

Curve Sketching

Draw a possible graph of $f(x)$ given the information below.

1. **a.** $f(x)$ is continuous

b. $f(3)=2$

c. $f'(x)>0,\ (-\infty,0),(3,\infty)$

d. $f'(x)<0,\ (0,3)$

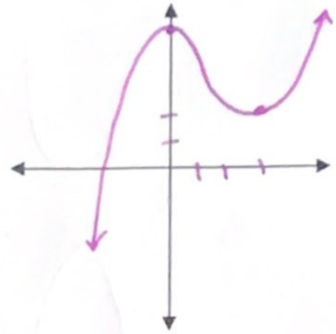
e. $f'(x)=0$ at x=0 ,x=3

$f(x)$: inc dec inc

$f'(x)$: + (0) − (3) +

0 Max, 3 Min

$f(x)$



2. **a.** $f(x)$ is continuous

b. $f'(x)<0,\ (-\infty,1)$

c. $f'(x)>0,\ (1,\infty)$

d. $f'(x)=$ undef. at x=1

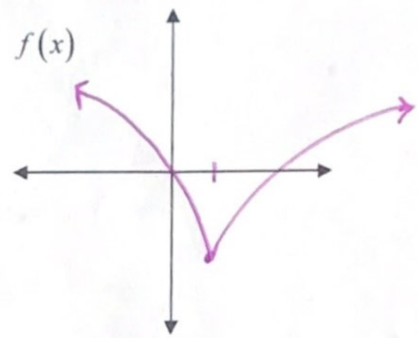
e. $f''(x)<0$ at $(-\infty,1)\cup(1,\infty)$

$f(x)$: dec conc down; inc conc up

$f'(x)$: − (und) +

$f''(x)$: − (und) −

$f(x)$



3. **a.** $f(x)$ is continuous

b. $f'(x)<0;(-\infty,2),(2,\infty)$

c. $f'(x)$ is undefined at $x=2$

d. $f''(x)<0$ when $x<2$

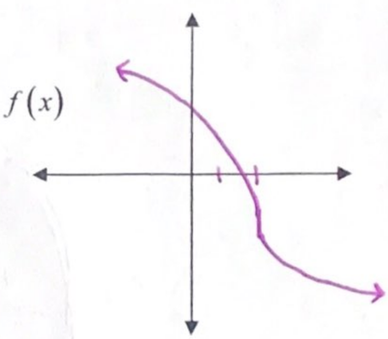
e. $f''(x)>0$ when $x>2$

$f(x)$: dec conc down; dec conc up

$f'(x)$: − VT (2) −

$f''(x)$: − (2) +

$f(x)$



4. **a.** $f(x)$ has jump discont. at $x=-2$

b. $f'(x)>0;(-\infty,-2),\cup(-2,\infty)$

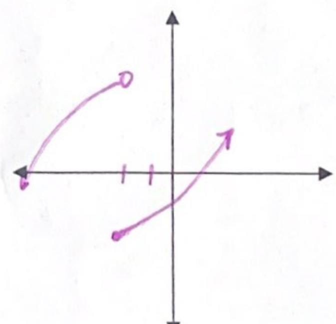
c. $f''(x)<0;(-\infty,-2)$

d. $f''(x)>0;(-2,\infty)$

$f(x)$: inc conc down; inc conc up

$f'(x)$: + jump (−2) +

$f''(x)$: − (−2) +



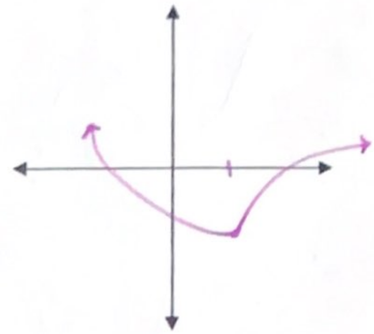
5.

a. $f(x)$ is continuous
b. $f'(x) < 0$ when $x < 1$
c. $f'(x) > 0$ when $x > 1$
d. $f''(x) > 0$ when $x < 1$
e. $f''(x) < 0$ when $x > 1$
f. $f'(x)$ does not exist at $x = 1$
g. $f''(x)$ does not exist at $x = 1$

$f(x)$ dec conc up | inc conc down $f(x)$

$f'(x)$: − DNE +

$f''(x)$: + DNE −



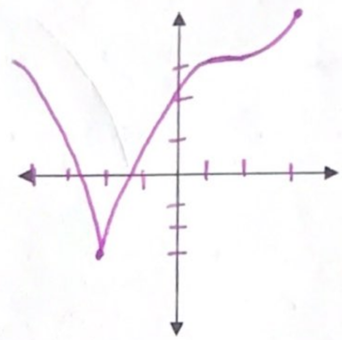
6.

a. $f(x)$ is continuous [-4, 3]
b. $f'(x) < 0$ on $(-4, -2)$
c. $f'(x) > 0$ on $(-2, 1) \cup (1, 3)$
d. $f'(x) =$ undef. at $x = -2$
e. $f(-2) = -3 \quad f(1) = 3$
f. $f'(x) = 0$ at x=1
g. $f'' < 0$ on $(-4, -2) \cup (-2, 1)$
h. $f'' > 0$ on (1, 3)

$f(x)$ dec conc down | inc conc down | inc conc up $f(x)$

$f'(x)$: −4 − cusp −2 + HT 1 + 3

$f''(x)$: −4 − −2 − 1 + 3



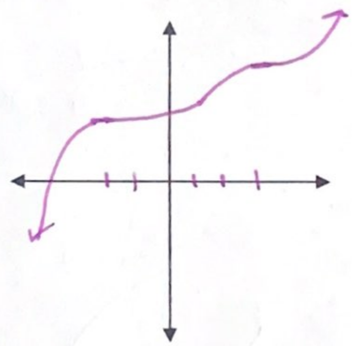
7.

a. $f(x)$ is continuous
b. $f'(x) > 0$ everywhere
c. $f'(x) = 0$ when $x = -2,\ x = 3$
d. $f''(x) < 0$ on $(-\infty, -2) \cup (1, 3)$
e. $f''(x) > 0$ on $(-2, 1) \cup (3, \infty)$

$f(x)$ | $f(x)$

$f'(x)$: + HT −2 + HT 3 +

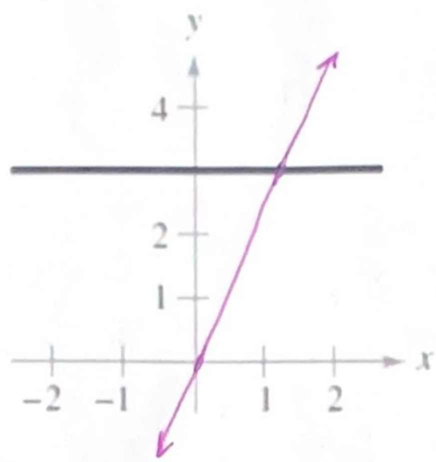
$f''(x)$: − −2 + 1 − 3 +



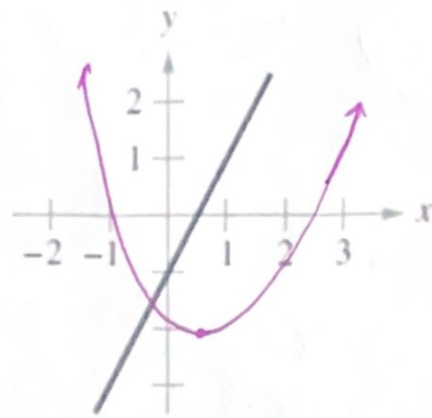
Curve Sketching - Graphing f from f'

The graph of f' is given below. Sketch a possible graph of f

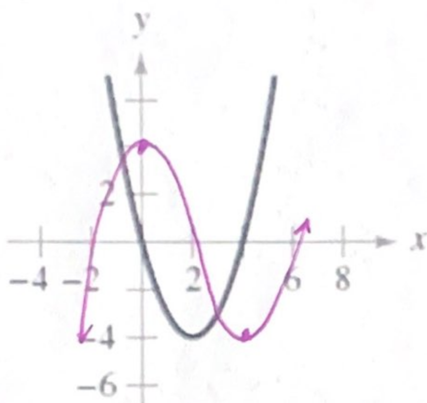
1.



2.



3.



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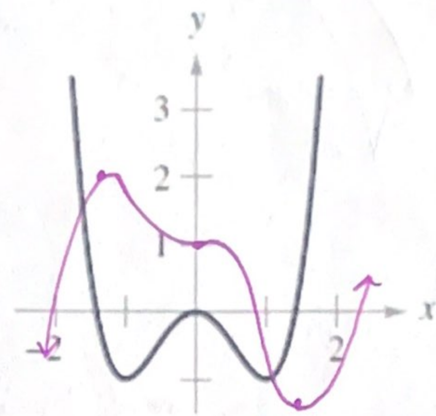

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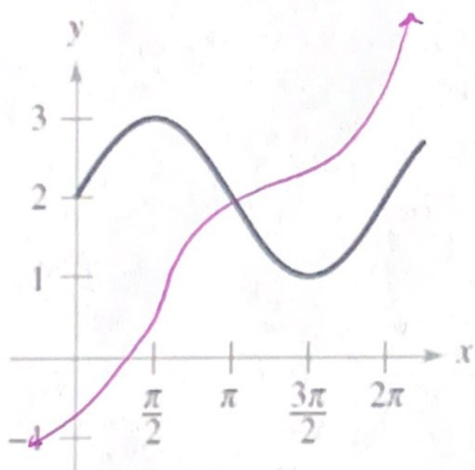

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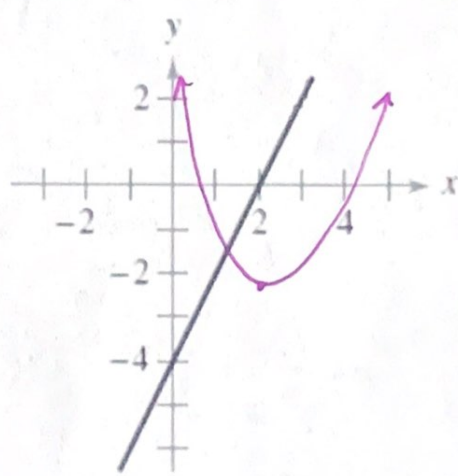

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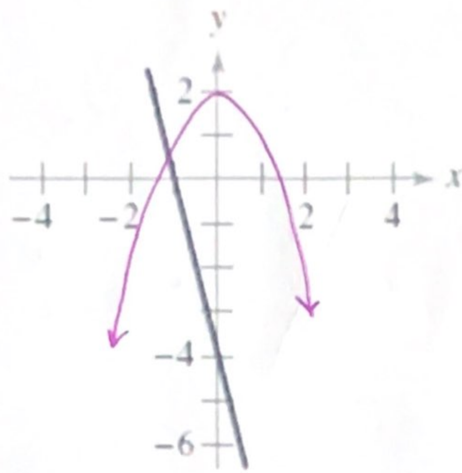

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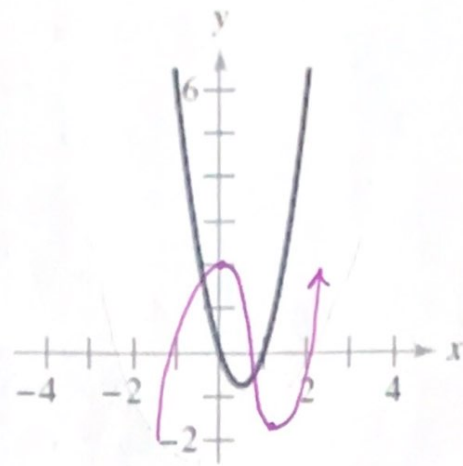
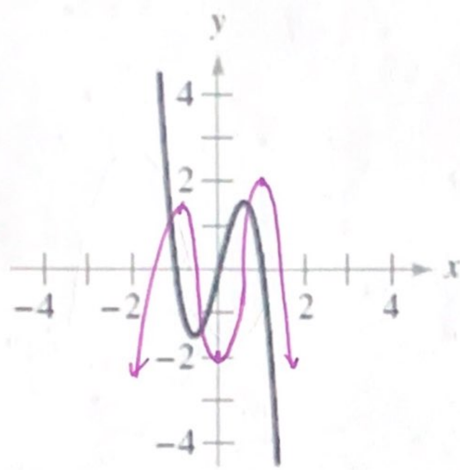
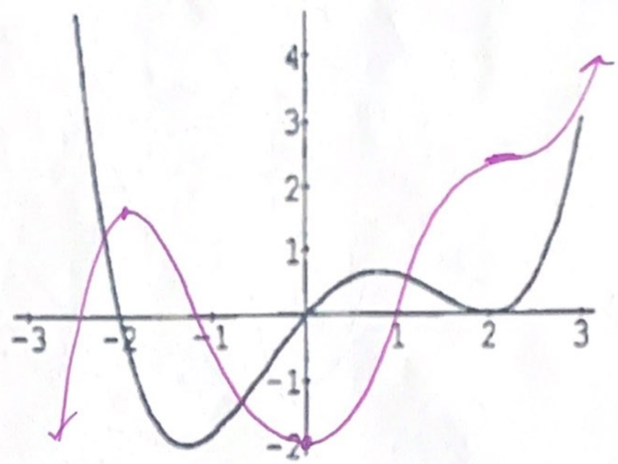


9.



10.



Practice Test – The Meaning of Derivatives

1. Multiple Choice: If f is a differentiable function, the $f'(a)$ is given by which of the following?

   I. $\lim_{h\to a} \frac{f(a+h)-f(a)}{h}$ × $h \to 0$

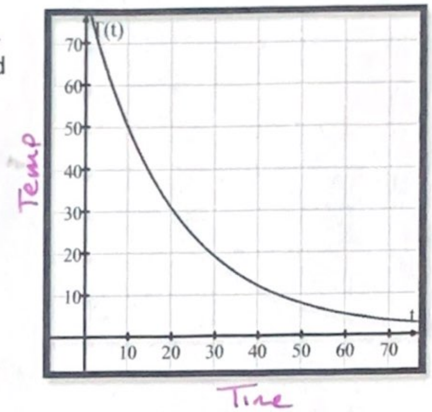
   II. $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ ✓

   III. $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ ✓

a. II and III only b. I and II only c. III only d. I and III only e. I, II, III

2. A pot of hot soup is placed in the freezer to cool down. The temperature, T, of the soup at time t is given by the graph, where T is measured in °F and t is measured in minutes. Estimate $T'(35)$ and interpret its meaning.

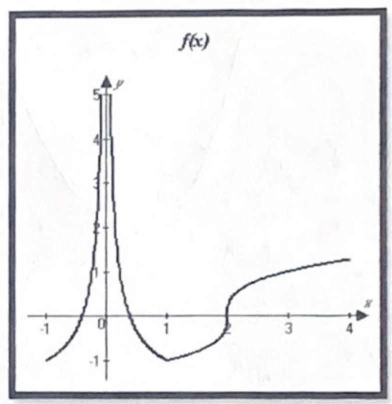
The instantaneous Rate of change in temp at 35 min given in degrees/min



3. Use the graph of f to determine all value(s) of x such that g is not differentiable. Give a reason for each answer.

$x = 0$ infinite disc

$x = 2$ vertical tangent



4. a. Find the derivative of $f(x) = \frac{4}{x-5}$ at $x = 7$. $f(7) = 2$

$$\lim_{x\to 7} \frac{\frac{4}{x-5} - 2}{x-7}$$

$$\lim_{x\to 7} \frac{4 - 2x + 10}{(x-5)(x-7)}$$

$$\lim_{x\to 7} \frac{-2}{x-5} = \frac{-2}{2} = -1$$

$f'(7) = -1$

b. Write the equation of the tangent line for part a.

$y - 2 = -1(x-7)$

5. a. Use the definition of derivative to find the derivative of f(x) = $x^2 - 3x + 2$.

$$\lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 2 - x^2 + 3x - 2}{h}$$

$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 2 - x^2 + 3x - 2}{h}$$

$$\lim_{h \to 0} \frac{2xh - h^2 - 3h}{h}$$

$$\lim_{h \to 0} 2x + h - 3$$

$$= 2x - 3$$

b. Find an equation of the tangent line to the curve at the point where x = -2.

$$f'(x) = 2x - 3$$

$$f'(-2) = 2(-2) - 3 = -7$$

$$f(-2) = (-2)^2 - 3(-2) + 2$$

$$= 4 + 6 + 2$$

$$= 12$$

$$y - 12 = -7(x+2)$$

6. Find the derivative as a function of x if f(x) = $\sqrt{5x+2}$.

$$\lim_{h \to 0} \frac{\sqrt{5(x+h)+2} - \sqrt{5x+2}}{h}$$

$$\lim_{h \to 0} \frac{5x + 5h + 2 - 5x - 2}{h\left(\sqrt{5(x+h)+2} + \sqrt{5x+2}\right)}$$

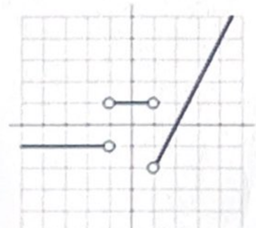
$$\lim_{h \to 0} \frac{5}{\sqrt{5(x+h)+2} + \sqrt{5x+2}}$$

$$= \frac{5}{\sqrt{5x+2} + \sqrt{5x+2}}$$

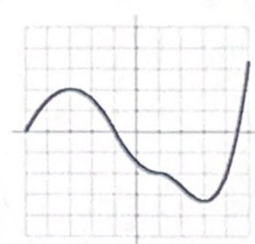
$$= \frac{5}{2\sqrt{5x+2}}$$

7. Below, in no particular order, are the graphs of $f(x)$, $f'(x)$, and $f''(x)$. Decide which graph goes with each function.

$f'(x)$



$f''(x)$



$f(x)$

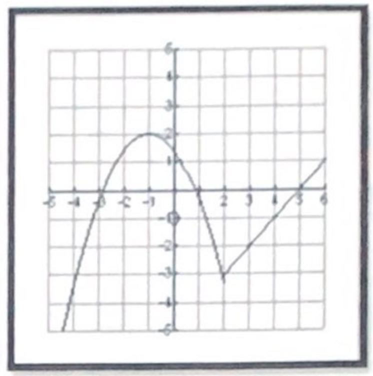
In #8 and 9, Multiple Choice.

8. The graph of a function $y = f(x)$ is shown below. Which of the following are true for the function?

   I. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$ ✓ $= -3$

   II. $f'(2)$ is defined × cusp

   III. $f'(x) < 0$ for all x in the open interval $(-5,2)$ ×



a. I only   b. II only   c. I and III   d. I and II   e. I, II, and III

9. $\lim_{h\to 0} \frac{\ln(e+h)-1}{h} =$

a. $f'(x)$ where $f(x) = \ln x$ at $x = 0$

b. $f'(x)$ where $f(x) = \frac{\ln x}{x}$ at $x = e$

c. $f'(x)$ where $f(x) = \ln x$ at $x = 1$

d. $f'(x)$ where $f(x) = \ln(x+e)$ at $x = 1$

e. $f'(x)$ where $f(x) = \ln x$ at $x = e$

$f(x) = \ln x$

$a = e$

10. The diameter D of a metal shaft, measured in cm, is recorded at various times t, measured in minutes, during a particular manufacturing process. Given the table of values below,

t	0	2	4	6	8	10
D	1.112	1.130	1.144	1.139	1.127	1.109

(a) Find the average rate of change over the interval [0, 2]. **Include units.**

$AROC_{[0,2]} = \frac{f(2)-f(0)}{2-0} = \frac{1.130-1.112}{2} = \frac{.018}{2} = .009$ cm/min

(b) Approximate the rate of change of D with respect to t when $t = 2$ minutes. **Include units.**

$D'(2) = \frac{AROC_{[0,2]} + AROC_{[2,4]}}{2} = \frac{.009 + \frac{1.144-1.130}{4-2}}{2}$

$= \frac{.009 + .007}{2}$

$= \frac{.016}{2}$

$= .008$ cm/min

11. The cost, C (in dollars) to produce g gallons of ice cream can be expressed as $C = f(g)$. Interpret the following in practical terms, giving units.

a) $f(400) = 580$ 400 gallons of ice cream cost \$580

b) $f'(100) = 2.9$ The IROC in cost when 100 gallons of ice cream is produced is 2.9 dollars per gallon

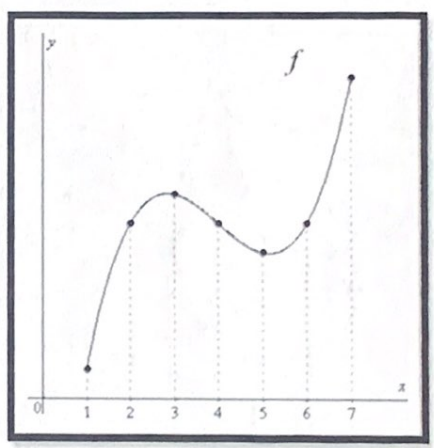
c) $f^{-1}(150) = 65$ It cost \$150 to produce 65 gallons of ice cream.

12. Suppose that the line tangent to the graph of $y = f(x)$ at $x = 4$ passes through the points $(-2, 6)$ and $(4, 3)$.

a) Find $f(4)$. $f(4) = 3$

b) Find $f'(4)$. $f'(4) = m_{tan} = \frac{3-6}{4+2} = \frac{-3}{6} = \frac{-1}{2}$

13. Use the graph of f to determine the interval(s) of x that meet the following conditions.



a. $f'(x) > 0$ $(1,3) \cup (5,7)$
inc

b. $f'(x) < 0$ $(3,5)$
dec

c. $f''(x) > 0$ $(4,7)$
conc up

d. $f''(x) < 0$ $(1,4)$
conc down

14. Given $f(4)=8$ and $f'(4)=3$, find the equations for the tangent line to $f(x)$ at $x=4$.

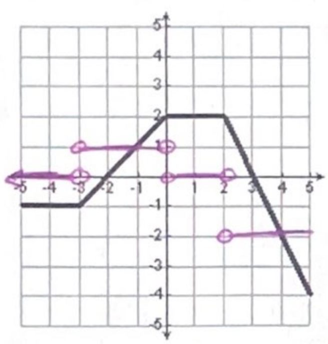
$y - 8 = 3(x-4$

15. Find the average rate of change of $f(x)=e^x+4$ over [0, 3].

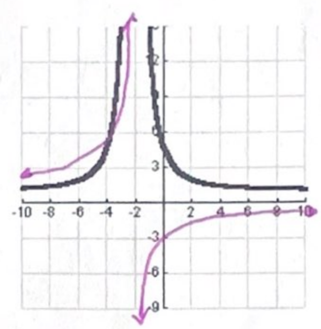
$$ARoC_{[0,3]} = \frac{f(3)-f(0)}{3-0}$$
$$= \frac{e^3+4-e^0-4}{3}$$
$$= \frac{e^3-1}{3}$$

Sketch the derivative of the following.

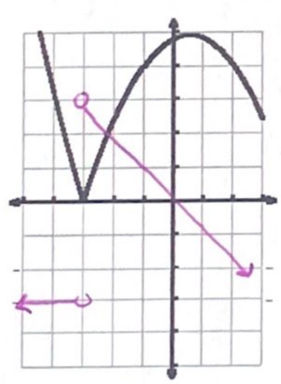
16.



17.



19.



20.