Unit 3 - Derivatives

- Notes and some practice are included
- Homework will be assigned on a daily basis

Topics Covered:

- Definition of a Derivative
- Tangent & Normal Lines
- Power Rule
- Product & Quotient Rule
- Chain Rule
- Derivatives of Trig Functions
- Derivatives of Exponential & Log Functions
- L'Hopital's Rule

Quiz is ____________________

Test is ____________________

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F19

Using the Definition of a Derivative

Use the definition of the derivative to find the derivative of each function with respect to x.

1. $y = -5x^2 - 2x + 5$

$$y' = \lim_{h\to 0} \frac{-5(x+h)^2 - 2(x+h) + 5 - (-5x^2 - 2x + 5)}{h}$$

$$\frac{-5(x^2 + 2xh + h^2) - 2x - 2h + 5 + 5x^2 + 2x - 5}{h}$$

$$\frac{\cancel{-5x^2} - 10xh - 5h^2 \cancel{-2x} - 2h + \cancel{5} + \cancel{5x^2} + \cancel{2x} \cancel{-5}}{h}$$

$$\frac{\cancel{h}(-10x - 5h - 2)}{\cancel{h}} \rightarrow \lim_{h\to 0} -10x - 5(0) - 2$$

$$y' = -10x - 2$$

2. $y = 2x - 1$

$$y' = \lim_{h\to 0} \frac{2(x+h) - 1 - (2x - 1)}{h}$$

$$= \frac{\cancel{2x} + 2\cancel{h} \cancel{-1} \cancel{-2x} \cancel{+1}}{\cancel{h}}$$

$$\lim_{h\to 0} 2$$

$$y' = 2$$

3. $y = -\frac{2}{x+4}$

$$y' = \lim_{h\to 0} \frac{\frac{-2}{x+h+4} + \left(\frac{+2}{x+4}\right)}{h}$$

$$\frac{\frac{-2(x+4)}{(x+h+4)(x+4)} + \frac{2(x+h+4)}{(x+h+4)(x+4)}}{h}$$

$$\frac{\cancel{-2x} \cancel{-8} + \cancel{2x} + 2\cancel{h} \cancel{+8}}{(x+h+4)(x+4)} \cdot \frac{1}{\cancel{h}}$$

$$\lim_{h\to 0} \frac{2}{(x+0+4)(x+4)} = y' = \frac{2}{(x+4)^2}$$

4. $f(x) = 2\sqrt{x+3}$

$$f'(x) = \frac{2\sqrt{x+h+3} - 2\sqrt{x+3}}{h} \cdot \frac{(2\sqrt{x+h+3} + 2\sqrt{x+3})}{(2\sqrt{x+h+3} + 2\sqrt{x+3})}$$

$$\frac{4(x+h+3) - 4(x+3)}{h(2\sqrt{x+h+3} + 2\sqrt{x+3})} = \frac{\cancel{4x} + 4\cancel{h} + \cancel{12} \cancel{-4x} \cancel{-12}}{\cancel{h}(2\sqrt{x+h+3} + 2\sqrt{x+3})}$$

$$\lim_{h\to 0} \frac{4}{2\sqrt{x+0+3} + 2\sqrt{x+3}} = \frac{4}{4\sqrt{x+3}}$$

$$f'(x) = \frac{1}{\sqrt{x+3}}$$

5. $\sqrt{2x - 5}$

$$\lim_{h\to 0} \frac{\sqrt{2(x+h)-5} - \sqrt{2x-5}}{h} \cdot \frac{(\sqrt{2(x+h)-5} + \sqrt{2x-5})}{(\sqrt{2(x+h)-5} + \sqrt{2x-5})}$$

$$= \frac{2(x+h) - 5 - (2x-5)}{h(\sqrt{2(x+h)-5} + \sqrt{2x-5})} = \frac{\cancel{2x} + 2\cancel{h} \cancel{-5} \cancel{-2x} \cancel{+5}}{\cancel{h}(\sqrt{2(x+h)-5} + \sqrt{2x-5})}$$

$$\lim_{h\to 0} \frac{2}{\sqrt{2(x+0)-5} + \sqrt{2x-5}}$$

$$f'(x) = \frac{2}{2\sqrt{2x-5}} \qquad f'(x) = \frac{1}{\sqrt{2x-5}}$$

6. $y = x^3$

$$y' = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$\frac{(x+h)(x^2 + 2xh + h^2) - x^3}{h}$$

$$\frac{\cancel{x^3} + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3 \cancel{-x^3}}{h}$$

$$\frac{3x^2h + 3xh^2 + h^3}{h} = \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$\lim_{h\to 0} 3x^2 + 3x(0) + (0)^2$$

$$y' = 3x^2$$

Tangent & Normal Lines

For each problem, find the equation of the tangent line to the function at the given point. Write your answer in point-slope form.

1. $f(x) = x^2 + 4x + 2$ at $(-1, -1)$

$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 + 4(x+h) + 2 - (x^2+4x+2)}{h}$

$\frac{x^2 + 2xh + h^2 + 4x + 4h + 2 - x^2 - 4x - 2}{h} = \frac{2xh + h^2 + 4h}{h}$

$\lim_{h \to 0} \frac{h(2x+h+4)}{h} = 2x + 0 + 4$

$f'(x) = 2x + 4$

$f'(-1) = 2(-1) + 4 \quad m = 2$

$y + 1 = 2(x+1)$ or $y = 2x + 1$

2. $f(x) = 2x^2 - 4$ at $(-1, -2)$

$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 - 4 - (2x^2 - 4)}{h}$

$\frac{2(x^2 + 2xh + h^2) - 4 - 2x^2 + 4}{h}$

$\frac{2x^2 + 4xh + 2h^2 - 4 - 2x^2 + 4}{h}$

$f'(x) = \lim_{h \to 0} 4x + 2(0) \quad f'(x) = 4x$

$f'(-1) = 4(-1)$

$f'(-1) = -4$

$m = -4$

$y + 2 = -4(x+1)$ or $y = -4x - 6$

3. $f(x) = \frac{1}{x-3}$ at $\left(0, -\frac{1}{3}\right)$

$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h-3} - \frac{1}{x-3}}{h} = \frac{\frac{x-3}{(x+h-3)(x-3)} - \frac{x+h-3}{(x+h-3)(x-3)}}{h}$

$\frac{\frac{x-3-x-h+3}{(x+h-3)(x-3)}}{h} = \frac{-h}{(x+h-3)(x-3)} \cdot \frac{1}{h}$

$\lim_{h \to 0} \frac{-1}{(x+0-3)(x-3)} = f'(x) = \frac{-1}{(x-3)^2}$

$f'(0) = \frac{-1}{(0-3)^2}$

$y + \frac{1}{3} = -\frac{1}{9}x$ or $y = -\frac{1}{9}x - \frac{1}{3}$ $\quad m = -\frac{1}{9}$

4. $f(x) = \sqrt{2x+2}$ at $(1, 2)$

$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{2(x+h)+2} - \sqrt{2x+2}\right)\left(\sqrt{2(x+h)+2} + \sqrt{2x+2}\right)}{h\left(\sqrt{2(x+h)+2} + \sqrt{2x+2}\right)}$

$\frac{2(x+h)+2 - (2x+2)}{h\left(\sqrt{2(x+h)+2} + \sqrt{2x+2}\right)} = \frac{2x + 2h + 2 - 2x - 2}{h\left(\sqrt{2(x+h)+2} + \sqrt{2x+2}\right)}$

$f'(x) = \lim_{h \to 0} \frac{2}{\sqrt{2(x+0)+2} + \sqrt{2x+2}}$

$f'(x) = \frac{2}{2\sqrt{2x+2}} = \frac{1}{\sqrt{2x+2}}$

$f'(1) = \frac{1}{\sqrt{2 \cdot 1 + 2}} = \frac{1}{2} = m$

$y - 2 = \frac{1}{2}(x-1)$ or $y = \frac{1}{2}x + \frac{3}{2}$

For each problem, find the equation of the normal line to the function at the given point. Write your answer in point-slope form.

perpendicular

5. $f(x) = \frac{4}{x}$ at $(-2, -2)$

$f'(x) = \frac{\frac{4}{x+h} - \frac{4}{x}}{h} = \frac{\frac{4x}{x(x+h)} + \frac{-4(x+h)}{x(x+h)}}{h}$

$\frac{4x - 4x - 4h}{x(x+h)} \cdot \frac{1}{h} \quad \lim_{h \to 0} \frac{-4}{x(x+h)}$

$f'(x) = \frac{-4}{x^2} \quad f'(-2) = \frac{-4}{(-2)^2} = -1$

$m = -1$

$\perp m = 1$

$y + 2 = 1(x+2)$

or $y = x$

6. $f(x) = \sqrt{x+3}$ at $(6, 3)$

$\lim_{h \to 0} \frac{\left(\sqrt{x+h+3} - \sqrt{x+3}\right)\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}$

$\frac{x+h+3 - (x+3)}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+0+3} + \sqrt{x+3}}$

$f'(x) = \frac{1}{2\sqrt{x+3}} \quad f'(6) = \frac{1}{2\sqrt{6+3}} = \frac{1}{6}$

$m = 1/6$

$\perp m = -6$

$y - 3 = -6(x-6)$

or $y = -6x + 39$

Constant and Power Rule Practice

Find the derivative of each function. Make sure your answers are simplified completely. If a point is given, find the value of the derivative at that point.

1. $y = 3$

$y' = 0$

2. $f(x) = x + 1$

$f'(x) = 1$

3. $f(t) = -3t^2 + 2t - 4$

$f'(t) = -6t + 2$

4. $s(t) = t^3 - 2t + 4$

$s'(t) = 3t^2 - 2$

5. $y = 4t^{\frac{4}{3}}$

$y' = \frac{16}{3} t^{1/3}$

or

$y' = \frac{16}{3}\sqrt[3]{t}$

6. $f(x) = 4\sqrt{x} = 4x^{1/2}$

$f'(x) = 2x^{-1/2} = \frac{2}{x^{1/2}}$

$f'(x) = \frac{2}{\sqrt{x}}$

7. $y = 4x^{-2} + 2x^2$

$y' = -8x^{-3} + 4x$

$y' = \frac{-8}{x^3} + 4x$

8. $y = \frac{1}{4x^3} = \frac{1}{4}x^{-3}$

$y' = -\frac{3}{4}x^{-4}$

$y' = \frac{-3}{4x^4}$

9. $y = \frac{1}{(4x)^3} = (4x)^{-3}$

$y = \frac{1}{64x^3} = \frac{1}{64}x^{-3}$

$y' = -\frac{3}{64}x^{-4}$

$y' = \frac{-3}{64x^4}$

10. $y = \frac{\sqrt{x}}{x} = \frac{x^{1/2}}{x} = x^{1/2 - 1} = x^{-1/2}$

$y' = -\frac{1}{2}x^{-3/2} = \frac{-1}{2x^{3/2}}$

$y' = \frac{-1}{2\sqrt{x^3}}$

11. $f(x) = x^2 - \frac{4}{x} = x^2 - 4x^{-1}$

$f'(x) = 2x + 4x^{-2}$

$f'(x) = 2x + \frac{4}{x^2}$

12. $f(x) = x^2 - 2x - \frac{2}{x^4} = x^2 - 2x - 2x^{-4}$

$f'(x) = 2x - 2 + 8x^{-5}$

$f'(x) = 2x - 2 + \frac{8}{x^5}$

3

13. $f(x)=\dfrac{2x^3-4x^2+3}{x^2}$

$f(x)=2x-4+3x^{-2}$

$f'(x)=2-6x^{-3}$

$f'(x)=2-\dfrac{6}{x^3}$

14. $y=x(x^2+1)$

$y=x^3+x$

$y'=3x^2+1$

15. $f(x)=x^{4/5}$

$f'(x)=\frac{4}{5}x^{-1/5}$

$f'(x)=\dfrac{4}{5x^{1/5}}$ or $\dfrac{4}{5\sqrt[5]{x}}$

16. $f(x)=\sqrt[3]{x}+\sqrt[5]{x}$

$f(x)=x^{1/3}+x^{1/5}$

$f'(x)=\frac{1}{3}x^{-2/3}+\frac{1}{5}x^{-4/5}$

$f'(x)=\dfrac{1}{3\sqrt[3]{x^2}}+\dfrac{1}{5\sqrt[5]{x^4}}$

17. $f(x)=\dfrac{4}{x^{-3}}=4x^3$

$f'(x)=12x^2$

18. $f(x)=\dfrac{\pi}{(3x)^2}=\dfrac{\pi}{9x^2}=\dfrac{\pi}{9}x^{-2}$

$f'(x)=-\dfrac{2\pi}{9}x^{-3}$

$f'(x)=-\dfrac{2\pi}{9x^3}$

19. $f(x)=\dfrac{1}{\sqrt[9]{x^7}}=\dfrac{1}{x^{7/9}}=x^{-7/9}$

$f'(x)=-\frac{7}{9}x^{-16/9}$

$f'(x)=\dfrac{-7}{9x^{16/9}}$ or $\dfrac{-7}{9\sqrt[9]{x^{16}}}$

20. $f(x)=(x^2+2x)(x+1)$

$f(x)=x^3+3x^2+2x$

$f'(x)=3x^2+6x+2$

21. $f(x)=\dfrac{5x^7+9x^4+2x-9}{10}$

$f(x)=\frac{1}{2}x^7+\frac{9}{10}x^4+\frac{1}{5}x-\frac{9}{10}$

$f'(x)=\frac{7}{2}x^6+\frac{18}{5}x^3+\frac{1}{5}$

22. $f(x)=(2x+1)^2$ $(0,1)$

$f(x)=4x^2+4x+1$

$f'(x)=8x+4$

$f'(0)=8(0)+4=4$

23. $f(x)=\dfrac{1}{x}$ at $\left(2,\dfrac{1}{2}\right)$

$f(x)=x^{-1}$

$f'(x)=-1x^{-2}$

$f'(x)=\dfrac{-1}{x^2}$

$f'(2)=\dfrac{-1}{(2)^2}$

$=-1/4$

4

24. $f(x)=x(x^2+1)$ $(7,350)$

$f(x)=x^3+x$

$f'(x)=3x^2+1$

$f'(7)=3(7)^2+1$

$f'(7)=148$

PRODUCT & QUOTIENT RULE

PRODUCT RULE: Find the derivative 2 different ways: 1) using the Product Rule and 2) distribute first & then derive.

1. $f(x) = x^2(x^3 - 1)$

$$f'(x) = x^2(3x^2) + (x^3-1)(2x)$$
$$= 3x^4 + 2x^4 - 2x$$
$$f'(x) = 5x^4 - 2x$$
$$f(x) = x^5 - x^2$$
$$f'(x) = 5x^4 - 2x$$

2. $f(x) = (x^3 - 1)(x^2 - 2x + 1)$

$$f'(x) = (x^3-1)(2x-2) + (x^2-2x+1)(3x^2)$$
$$f'(x) = 2x^4 - 2x^3 - 2x + 2 + 3x^4 - 6x^3 + 3x^2$$
$$f'(x) = 5x^4 - 8x^3 + 3x^2 - 2x + 2$$
$$f(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$$
$$f'(x) = 5x^4 - 8x^3 + 3x^2 - 2x + 2$$

Differentiate with respect to x using the Product Rule.

3. $f(x) = (4x + 1)(3x - 2)$

$$f'(x) = (4x+1)(3) + (3x-2)(4)$$
$$f'(x) = 12x + 3 + 12x - 8$$
$$f'(x) = 24x - 5$$

4. $f(x) = (x + 3)^2$

$$f(x) = (x+3)(x+3)$$
$$f(x) = x^2 + 6x + 9$$
$$f'(x) = 2x + 6$$

5. $f(x) = (2x^{\frac{5}{3}} - 3)(-x^5 + 3)$

$$f(x) = -2x^{20/3} + 6x^{5/3} + 3x^5 - 9$$
$$f'(x) = -\frac{40}{3}x^{17/3} + \frac{30}{3}x^{2/3} + 15x^4$$
$$f'(x) = -\frac{40}{3}x^{17/3} + 10x^{2/3} + 15x^4 \text{ or } \frac{-40\sqrt[3]{x^{17}}}{3} + 10\sqrt[3]{x^2} + 15x^4$$

6. $f(x) = (1 + \sqrt{x})(x^3) = (1 + x^{1/2})(x^3)$

$$f(x) = x^3 + x^{7/2}$$
$$f'(x) = 3x^2 + \frac{7}{2}x^{5/2}$$
$$f'(x) = 3x^2 + \frac{7\sqrt{x^5}}{2}$$

When dividing by a monomial, you don't need to use the Quotient Rule. You can simplify first and then derive. Find the derivative <u>without</u> using the Quotient Rule.

7. $f(x) = \frac{x^2+3x}{6} = \frac{x^2}{6} + \frac{3x}{6} = \frac{1}{6}x^2 + \frac{1}{2}x$

$$f'(x) = \frac{2}{6}x + \frac{1}{2}$$
$$f'(x) = \frac{1}{3}x + \frac{1}{2}$$

8. $f(x) = \frac{5x^4 - 16x^2 + 4}{x^2} = \frac{5x^4}{x^2} - \frac{16x^2}{x^2} + \frac{4}{x^2}$

$$f(x) = 5x^2 - 16 + 4x^{-2}$$
$$f'(x) = 10x - 8x^{-3}$$
$$f'(x) = 10x - \frac{8}{x^3}$$

9. $f(x) = \frac{-3(3x - 2x^2)}{7x} = \frac{-9x + 6x^2}{7x} = \frac{-9x}{7x} + \frac{6x^2}{7x}$

$$f(x) = -\frac{9}{7} + \frac{6}{7}x$$
$$f'(x) = 6/7$$

Differentiate with respect to x using the Quotient Rule.

10. $f(x) = \frac{x^2}{1-x^3}$

$f'(x) = \frac{(1-x^3)(2x) - x^2(-3x^2)}{(1-x^3)^2} = \frac{2x-2x^4+3x^4}{(1-x^3)^2}$

$f'(x) = \frac{x^4+2x}{(1-x^3)^2}$

11. $f(x) = \frac{x^2-2}{x^2+2}$

$f'(x) = \frac{(x^2+2)(2x) - (x^2-2)(2x)}{(x^2+2)^2}$

$f'(x) = \frac{2x^3+4x-2x^3+4x}{(x^2+2)^2} = \frac{8x}{(x^2+2)^2}$

12. $f(x) = \frac{x^3+3x+2}{x^2-1}$

$f'(x) = \frac{(x^2-1)(3x^2+3) - (x^3+3x+2)(2x)}{(x^2-1)^2}$

$f'(x) = \frac{3x^4+3x^2-3x^2-3-2x^4-6x^2-4x}{(x^2-1)^2}$

$f'(x) = \frac{x^4-6x^2-4x-3}{(x^2-1)^2}$

13. $f(x) = (2x-5)(4-x)^{-1} = \frac{2x-5}{4-x}$

$f'(x) = \frac{(4-x)(2) - (2x-5)(-1)}{(4-x)^2}$

$f'(x) = \frac{8-2x+2x-5}{(4-x)^2}$

$f'(x) = \frac{3}{(4-x)^2}$

Differentiate with respect to x using the Product or Quotient Rule.

14. $g(x) = \frac{3x^4+5x^2}{x^5+2}$

$g'(x) = \frac{(x^5+2)(12x^3+10x) - (3x^4+5x^2)(5x^4)}{(x^5+2)^2}$

$g'(x) = \frac{12x^8+10x^6+24x^3+20x-15x^8-25x^6}{(x^5+2)^2}$

$g'(x) = \frac{-3x^8-15x^6+24x^3+20x}{(x^5+2)^2}$

15. $y = \frac{\sqrt{x}}{x^3+1} = \frac{x^{1/2}}{x^3+1}$

$y' = \frac{(x^3+1)(\frac{1}{2}x^{-1/2}) - x^{1/2}(3x^2)}{(x^3+1)^2}$

$y' = \frac{\frac{x^3+1}{2\sqrt{x}} - 3x^{5/2}}{(x^3+1)^2}$

16. $m(x) = (-4 + \frac{4}{x})(3x^3 + 4)$

$m'(x) = (-4+\frac{4}{x})(9x^2) + (3x^3+4)(-\frac{4}{x^2})$

$m'(x) = -36x^2 + 36x - 12x - \frac{16}{x^2}$

$m'(x) = -36x^2 + 24x - \frac{16}{x^2}$

17. $h(x) = \frac{2x^4-5x^2+6x}{2x^2} = \frac{2x^4}{2x^2} - \frac{5x^2}{2x^2} + \frac{6x}{2x^2}$

$h(x) = x^2 - \frac{5}{2} + 3x^{-1}$

$h'(x) = 2x - 3x^{-2}$

$h'(x) = 2x - \frac{3}{x^2}$

18. $f(x) = (2x^3 - 5)(3x^2 + 4x - 7)$

$f'(x) = (2x^3-5)(6x+4) + (3x^2+4x-7)(6x^2)$

$f'(x) = 12x^4 + 8x^3 - 30x - 20 + 18x^4 + 24x^3 - 42x^2$

$f'(x) = 30x^4 + 32x^3 - 42x^2 - 30x - 20$

19. $y = 5\sqrt{x}(3x^2 + 4x - \sqrt{x})$

$y = 5x^{1/2}(3x^2 + 4x - x^{1/2})$

$y' = 5x^{1/2}(6x + 4 - \frac{1}{2x^{1/2}}) + (3x^2+4x-x^{1/2})(\frac{5}{2}x^{-1/2})$

$y' = 30x^{3/2} + 20x^{1/2} - \frac{5}{2} + \frac{15}{2}x^{3/2} + 10x^{1/2} - \frac{5}{2}$

$y' = \frac{75}{2}x^{3/2} + 30x^{1/2} - \frac{10}{2}$

or $y' = \frac{75\sqrt{x^3}}{2} + 30\sqrt{x} - 5$

Mixed Derivatives #1

Find the derivative using the power, product, or quotient rule. If necessary, rewrite first.

1. $y = 6x^3 + 4x^2 - 2x + 5$

$y' = 18x^2 + 8x - 2$

2. $y = \sqrt[4]{x^3} = x^{3/4}$

$y' = \frac{3}{4}x^{-1/4}$ $\quad y' = \frac{3}{4\sqrt[4]{x}}$

3. $y = 3x^2 + \frac{12}{\sqrt{x}} - \frac{1}{x^2} = 3x^2 + 12x^{-1/2} - x^{-2}$

$y' = 6x - 6x^{-3/2} + 2x^{-3}$

$y' = 6x - \frac{6}{\sqrt{x^3}} + \frac{2}{x^3}$ or $6x - \frac{6}{x^{3/2}} + \frac{2}{x^3}$

4. $y = 3 - 7x^3 + 3x^7$

$y' = -21x^2 + 21x^6$

5. $y = 3x^{\frac{2}{3}} + x^{\frac{3}{4}}$

$y' = -2x^{-5/3} + \frac{3}{4}x^{-1/4}$

$y' = \frac{-2}{x^{5/3}} + \frac{3}{4x^{1/4}}$

6. $y = \frac{3x^3 - 5}{7} = \frac{3}{7}x^3 - \frac{5}{7}$

$y' = \frac{9}{7}x^2$

7. $y = \frac{4x^{\frac{3}{2}}}{x} = 4x^{1/2}$

$y' = 2x^{-1/2}$ $\quad y' = \frac{2}{x^{1/2}}$ or $\frac{2}{\sqrt{x}}$

8. $y = \frac{x^2 + 1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + x^{-1}$

$y' = 1 - x^{-2}$

$y' = 1 - \frac{1}{x^2}$

9. $y = \frac{x^7 + 5x^6 - x^3}{x^2} = x^5 + 5x^4 - x$

$y' = 5x^4 + 20x^3 - 1$

10. $y = \frac{x+1}{\sqrt{x}} = \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}} = x^{1/2} + x^{-1/2}$

$y' = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$ $\quad y' = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

11. $y = (x^3 - 2)^2 = x^6 - 4x^3 + 4$

$y' = 6x^5 - 12x^2$

12. $y = \frac{x^2 - 4}{x + 3}$

$y' = \frac{(x+3)(2x) - (x^2-4)(1)}{(x+3)^2} = \frac{2x^2 + 6x - x^2 + 4}{(x+3)^2}$

$y' = \frac{x^2 + 6x + 4}{(x+3)^2}$

13. $y = \frac{2x+1}{2x-1}$

$y' = \frac{(2x-1)(2) - (2x+1)(2)}{(2x-1)^2} = \frac{4x - 2 - 4x - 2}{(2x-1)^2}$

$y' = \frac{-4}{(2x-1)^2}$

14. $y = \frac{x^2 + 1}{x^2 - 1}$

$y' = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}$

$y' = \frac{-4x}{(x^2-1)^2}$

15. $y = \frac{1}{1 + \sqrt{x}}$

$y' = \frac{(1+\sqrt{x})(0) - 1(\frac{1}{2\sqrt{x}})}{(1+\sqrt{x})^2}$ $\quad y' = \frac{-\frac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2}$

$y' = \frac{-1}{2\sqrt{x}(1+\sqrt{x})^2}$

16. $y = \frac{(x+1)(2x-5)}{(x+2)}$

$y = \frac{2x^2 - 3x - 5}{x + 2}$

$y' = \frac{(x+2)(4x-3) - (2x^2-3x-5)(1)}{(x+2)^2}$

$y' = \frac{4x^2 + 5x - 6 - 2x^2 + 3x + 5}{(x+2)^2}$

$y' = \frac{2x^2 + 8x - 1}{(x+2)^2}$

17. $y = (3x^3 + 4x)(x - 5)(x + 1)$

$y = (3x^3 + 4x)(x^2 - 4x - 5)$

$y = 3x^5 - 12x^4 - 15x^3 + 4x^3 - 16x^2 - 20x$

$y = 3x^5 - 12x^4 - 11x^3 - 16x^2 - 20x$

$y' = 15x^4 - 48x^3 - 33x^2 - 32x - 20$

7

Mixed Derivatives #2

Find the derivative using the <u>power</u>, <u>product</u>, or <u>quotient rule</u>. Remember to rewrite if necessary.

1. $y=\sqrt{x}+\frac{1}{\sqrt{x}}$ $y=x^{1/2}+x^{-1/2}$

$y'=\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}$

$y'=\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x^3}}$

2. $y=\sqrt{x}\left(\sqrt{x}+3\right)$

$y=x+3\sqrt{x}$ or $x+3x^{1/2}$

$y'=1+\frac{3}{2}x^{-1/2}$

$y'=1+\frac{3}{2\sqrt{x}}$

3. $y=x\left(x^2+\sqrt{x}\right)=x(x^2+x^{1/2})$

$y=x^3+x^{3/2}$

$y'=3x^2+\frac{3}{2}x^{1/2}$

$y'=3x^2+\frac{3\sqrt{x}}{2}$

4. $y=\frac{x^4-x^3}{x^{\frac{1}{2}}}=\frac{x^4}{x^{1/2}}-\frac{x^3}{x^{1/2}}$

$y=x^{7/2}-x^{5/2}$

$y'=\frac{7}{2}x^{5/2}-\frac{5}{2}x^{3/2}$

or $y'=\frac{7\sqrt{x^5}-5\sqrt{x^3}}{2}$

5. $y=\frac{4x}{3x+1}$

$y'=\frac{(3x+1)(4)-4x(3)}{(3x+1)^2}=\frac{12x+4-12x}{(3x+1)^2}$

$y'=\frac{4}{(3x+1)^2}$

6. $y=\sqrt[3]{x^2}-\sqrt[3]{x}$

$y=x^{2/3}-x^{1/3}$

$y'=\frac{2}{3}x^{-1/3}-\frac{1}{3}x^{-2/3}$

$y'=\frac{2}{3\sqrt[3]{x}}-\frac{1}{3\sqrt[3]{x^2}}$

7. $y=(3x-2)(x^3+1)$

$y=3x^4-2x^3+3x-2$

$y'=12x^3-6x^2+3$

8. $y=x^3(2x^4-x)$

$y'=x^3(8x^3-1)+(2x^4-x)(3x^2)$

$y'=8x^6-x^3+6x^6-3x^3$

$y'=14x^6-4x^3$

9. $y=\frac{3x+4}{2x-3}$

$y=\frac{(2x-3)(3)-(3x-4)(2)}{(2x-3)^2}=\frac{6x-9-6x-8}{(2x-3)^2}$

$y'=\frac{-17}{(2x-3)^2}$

10. $y=\frac{x^3-x^2+2}{x^2}$ $y=\frac{x^3}{x^2}-\frac{x^2}{x^2}+\frac{2}{x^2}$

$y=x-1+2x^{-2}$

$y'=1-4x^{-3}$

$y'=1-\frac{4}{x^3}$

11. Find <u>all</u> derivatives:

$y=x^5+\frac{1}{6}x^2-\frac{1}{3}x$

$y'=5x^4+\frac{1}{3}x-\frac{1}{3}$

$y''=20x^3+\frac{1}{3}$

$y'''=60x^2$

$y^{IV}=120x$

$y^{V}=120$

$y^{VI}=0$

12. Given: $y=x^3-5x^2+3x-1$. $(2)^3-5(2)^2+3(2)-1$

$8-20+6-1=-7$

Write the equation of the <u>tangent</u> line to the curve at $x=2$. (Hint: find y' first.)

$(2,-7)$

$y'=3x^2-10x+3$

$3(2)^2-10(2)+3=-5$ $m=-5$

$y+7=-5(x-2)$

(8)

Chain Rule

1. $y = (x^3 - 4)^4$

$y' = 4(x^3-4)^3(3x^2)$

$y' = 12x^2(x^3-4)^3$

2. $y = (2x^2 + 5)^7$

$y' = 7(2x^2+5)^6(4x)$

$y' = 28x(2x^2+5)^6$

3. $f(x) = (x^2 + 2x + 5)^6$

$f'(x) = 6(x^2+2x+5)^5(2x+2)$

$f'(x) = (12x+12)(x^2+2x+5)^5$

4. $f(x) = \sqrt[3]{x^2 + x}$ $= (x^2+x)^{1/3}$

$f'(x) = \frac{1}{3}(x^2+x)^{-2/3}(2x+1)$

$f'(x) = \dfrac{2x+1}{3(x^2+x)^{2/3}}$

5. $y = \sqrt{(3x + 1)^3}$ $= (3x+1)^{3/2}$

$y' = \frac{3}{2}(3x+1)^{1/2} \cdot 3$

$y' = \dfrac{9\sqrt{3x+1}}{2}$

6. $y = (\sqrt{x} + 1)^2$

$y' = 2(\sqrt{x}+1)\left(\frac{1}{2\sqrt{x}}\right)$

$y' = \dfrac{\sqrt{x}+1}{\sqrt{x}} = \dfrac{\sqrt{x}}{\sqrt{x}} + \dfrac{1}{\sqrt{x}}$

$y' = 1 + \dfrac{1}{\sqrt{x}}$

7. $g(x) = \dfrac{1}{\sqrt{2x^3 - 7x^2}}$

$g(x) = (2x^3-7x^2)^{-1/2}$

$g'(x) = -\frac{1}{2}(2x^3-7x^2)^{-3/2}(6x^2-14x)$

$g'(x) = \dfrac{-3x^2+7x}{\sqrt{(2x^3-7x^2)^3}}$

8. $y = (5x^2 - 3x)^{-\frac{2}{3}}$

$y' = -\frac{2}{3}(5x^2-3x)^{-5/3}(10x-3)$

$y' = \dfrac{-2(10x-3)}{3\sqrt[3]{(5x^2-3x)^5}}$

9. $y = \sqrt[3]{(x^2 + 4)^2}$

$y = (x^2+4)^{2/3}$

$y' = \frac{2}{3}(x^2+4)^{-1/3}(2x)$

$y' = \dfrac{4x}{3\sqrt[3]{x^2+4}}$

10. $f(x) = \dfrac{5}{(4x - 3)^2}$ $= 5(4x-3)^{-2}$

$f'(x) = -10(4x-3)^{-3}(4)$

$f'(x) = \dfrac{-40}{(4x-3)^3}$

11. $f(x) = (x^2 - 3)(5x - 1)^6$

$f'(x) = (x^2-3) \cdot 6(5x-1)^5 \cdot 5 + (5x-1)^6(2x)$

$f'(x) = 30(x^2-3)(5x-1)^5 + 2x(5x-1)^6$

12. $y = \sqrt{3x^2 + 5}$

$y = (3x^2+5)^{1/2}$

$y' = \frac{1}{2}(3x^2+5)^{-1/2}(6x)$

$y' = \dfrac{3x}{\sqrt{3x^2+5}}$

Derivatives of Trigonometric Functions

1. $y = 2\sin x$

$y' = 2\cos x$

2. $y = \frac{\sin x}{2}$

$y' = \frac{1}{2}\cos x$

or

$y' = \frac{\cos x}{2}$

3. $y = x + \cos x$

$y' = 1 - \sin x$

4. $y = x^2 - \frac{1}{2}\cos x$

$y' = 2x + \frac{1}{2}\sin x$

5. $y = 5 + \sin x$

$y' = \cos x$

6. $y = \frac{1}{x} - 3\sin x$ $\quad x^{-1} \quad \frac{-1}{x^2}$

$y' = \frac{-1}{x^2} - 3\cos x$

7. $y = \pi\cos x$

$y' = -\pi\sin x$

8. $y = 1 + x - \cos x$

$y' = 1 + \sin x$

9. $y = x^{-1} + 5\sin x$

$y' = \frac{-1}{x^2} + 5\cos x$

10. $y = \csc x - 5x + 7$

$y' = -\csc x \cot x - 5$

11. $y = 4\sqrt{x} + 3\cos x$

$y = 4x^{1/2} + 3\cos x$

$y' = 2x^{-1/2} - 3\sin x$

$y' = \frac{2}{\sqrt{x}} - 3\sin x$

12. $y = 2\sin x + 3\cos x$

$y' = 2\cos x - 3\sin x$

13. $y = x \cdot \cos x$

$y' = x \cdot -\sin x + \cos x$

$y' = -x\sin x + \cos x$

14. $y = 2\sin x - \tan x$

$y' = 2\cos x - \sec^2 x$

15. $y = 2x + \cot x$

$y' = 2 - \csc^2 x$

16. $y = \cot x \cdot \sec x$ Simplify 1st with Trig identities

$y = \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x}$

$y = \frac{1}{\sin x}$

$y = \csc x$

$y' = -\csc x \cot x$

17. $y = x^2 \sin x$

$y' = x^2 \cos x + 2x \sin x$

18. $y = 5 + \frac{1}{\tan x}$ Trig identity

$y = 5 + \cot x$

$y' = -\csc^2 x$

19. $y = \frac{\sin x}{x}$

$\frac{x \cdot \cos x - \sin x}{x^2}$

20. $y = \sin x \cdot \cos x$

$y' = \sin x(-\sin x) + \cos x \cos x$

$y' = -\sin^2 x + \cos^2 x$

21. $y = \sin x \cdot \sec x$ Simplify 1st

$y = \sin x \cdot \frac{1}{\cos x}$

$y = \frac{\sin x}{\cos x}$ or $\tan x$

$y' = \sec^2 x$

22. $y = \tan x \cdot \cot x$ Trig Reciprocal Identity

$y = 1$

$y' = 0$

23. $y = \frac{4}{\cos x} = 4 \cdot \frac{1}{\cos x}$

$y = 4 \sec x$

$y' = 4 \sec x \tan x$

24. $y = \sin(3x + 1)$

$y' = \cos(3x+1) \cdot 3$

$y' = 3\cos(3x+1)$

25. $y = \tan(2x - x^3)$

$y' = \sec^2(2x - x^3)(2 - 3x^2)$

$y' = (2 - 3x^2)\sec^2(2x - x^3)$

26. $y = \cos\left(-\frac{x}{3}\right)$

$y' = -\sin\left(\frac{-x}{3}\right) \cdot \left(\frac{-1}{3}\right)$

$y' = \frac{1}{3}\sin\left(\frac{-x}{3}\right)$

27. $y = 4\sin^2 x + 5\cos^2 x$

$y = 4(\sin x)^2 + 5(\cos x)^2$

$y' = 8\sin x \cdot \cos x + 10\cos x \cdot -\sin x$

$y' = 8\sin x \cos x - 10\sin x \cos x$

$y' = -2\sin x \cos x$

28. $y = \frac{1 + \tan^2 x}{\sec x}$ Pythagorean identity

$y = \frac{\sec^2 x}{\sec x}$

$y = \sec x$

$y' = \sec x \tan x$

29. $y = (1 + \cos 3x)^2$

$y' = 2(1 + \cos 3x)(-\sin 3x)(3)$

$y' = -6\sin(3x)(1 + \cos 3x)$

30. $f(x) = \cot\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)$ use quotient identity

$f(x) = \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot \sin\left(\frac{x}{2}\right)$

$f'(x) = -\sin\left(\frac{x}{2}\right) \cdot \frac{1}{2}$

$f'(x) = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$

Derivatives of e^x and a^x

Find the derivative of each.

1. $y = e^{2x}$

$y' = e^{2x} \cdot \ln e \cdot 2$

$y' = 2e^{2x}$

2. $y = e^{5x^2}$

$y' = e^{5x^2} \cdot \ln e \cdot 10x$

$y' = 10xe^{5x^2}$

3. $y = e^{\sin x}$

$y' = e^{\sin x} \cdot \ln e \cdot \cos x$

$y' = \cos x \cdot e^{\sin x}$

4. $y = e^{\tan x}$

$y' = e^{\tan x} \cdot \sec^2 x$

5. $y = e^{x^2+2x}$

$y' = e^{x^2+2x}(2x+2)$

6. $y = e^{\sqrt{x}}$

$y' = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$

$y' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

7. $y = 5^x$

$y' = 5^x \cdot \ln 5 \cdot 1$

$y' = 5^x \ln 5$

8. $y = e^{e^x}$

$y' = e^{e^x} \cdot \ln e \cdot e^x \cdot 1$

$y' = e^{e^x} \cdot e^x$

9. $y = 7^{x^2+2x^3}$

$y' = 7^{x^2+2x^3} \cdot \ln 7 \cdot (2x + 6x^2)$

10. $y = \sin e^{3x}$ chain

$y' = \cos(e^{3x})\, e^{3x} \cdot 3$

$y' = 3\cos(e^{3x})\, e^{3x}$

11. $y = xe^x$ product

$y' = xe^x + e^x \cdot 1$

$y' = xe^x + e^x$

or $y' = e^x(x+1)$

12. $y = (\sin x)e^x$ product

$y' = (\sin x)e^x + e^x(\cos x)$

or $y' = e^x(\sin x + \cos x)$

13. $y = x^2e^x$ product

$y' = x^2 \cdot e^x + e^x \cdot 2x$

$y' = x^2e^x + 2xe^x$

or $y' = xe^x(x+2)$

14. $y = \frac{e^x}{x^2}$ quotient

$y' = \frac{x^2e^x - e^x(2x)}{(x^2)^2}$

$y' = \frac{xe^x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}$

15. $y = 2^x(x^2+1)$ product

$y' = 2^x(2x) + (x^2+1)2^x \ln 2$

or

$y' = 2^x(2x + \ln 2 \cdot (x^2+1))$

16. $y = 3^{\ln x}$

$y' = 3^{\ln x} \ln 3 \cdot \frac{1}{x}$

$y' = \frac{3^{\ln x} \ln 3}{x}$

17. $y = x^2 + 4^x$

$y' = 2x + 4^x \ln 4$

18. $y = \ln e^{x^2}$ Rewrite with prop. of logs

$y = x^2 \ln e$

$y = x^2$

$y' = 2x$

19. $y = e^{\ln x^3}$ Rewrite 1st

$y = \ln e\, x^3$

$y = x^3$

$y' = 3x^2$

20. $y = e^{3x} \cdot 4^{5x}$

$y' = e^{3x} \cdot 4^{5x} \ln 4 \cdot 5 + 4^{5x} \cdot e^{3x} \cdot 3$

$y' = e^{3x} 4^{5x}(5\ln 4 + 3)$

21. $y = e^{\csc x}$

$y' = e^{\csc x} \cdot \ln e \cdot -\csc x \cot x$

$y' = -e^{\csc x} \csc x \cot x$

22. $y = 10^{\sin x}$

$y' = 10^{\sin x} \cos x \ln 10$

23. $y = x^2e^x - xe^x$ Factor out e^x

$y = e^x(x^2 - x)$

$y' = e^x(2x-1) + (x^2-x)e^x$

$y' = e^x(2x - 1 + x^2 - x)$

$y' = e^x(x^2 + x - 1)$

12

or $e^x x^2 + e^x x - e^x$

24. $y = xe^2 - e^x$

constant

$y = e^2x - e^x$

$y' = e^2 - e^x$

Derivatives of $\ln x$

Find each Derivative

1. $y = \ln(x^3 + 1)$ Chain

$y' = \frac{1}{x^3+1} \cdot 3x^2$

$y' = \frac{3x^2}{x^3+1}$

2. $y = \ln\sqrt{x}$ Chain

$y' = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$

$y' = \frac{1}{2x}$

3. $y = \sqrt{\ln(x)} = (\ln(x))^{1/2}$ chain

$\frac{1}{2}(\ln x)^{-1/2} \frac{1}{x}$

$y' = \frac{1}{2x\sqrt{\ln x}}$

4. $y = \ln|\sin x|$ Chain

$y' = \frac{1}{\sin x} \cdot \cos x$

$y' = \cot x$

5. $y = \ln(\sec x)$ Chain

$y' = \frac{1}{\sec x} \cdot \sec x \tan x$

$y' = \tan x$

6. $y = x \cdot \ln x$ Product

$y' = x \cdot \frac{1}{x} + \ln x \cdot 1$

$y' = 1 + \ln x$

7. $y = \frac{\ln x}{x^2}$ Quotient

$y' = \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{(x^2)^2}$

$y' = \frac{x - 2x\ln x}{x^4}$ or $\frac{1 - 2\ln x}{x^3}$

8. $y = \ln(\ln x)$ Chain

$y' = \frac{1}{\ln x} \cdot \frac{1}{x}$

$y' = \frac{1}{x\ln x}$

9. $y = (\sin x)(\ln x)$ Product

$y' = \sin x \cdot \frac{1}{x} + \ln x \cdot \cos x$

$y' = \frac{\sin x}{x} + \ln x \cos x$

10. $y = \frac{x^2}{\ln x}$ Quotient

$y' = \frac{\ln x \cdot 2x - x^2 \cdot \frac{1}{x}}{(\ln x)^2}$

$y' = \frac{2x\ln x - x}{(\ln x)^2}$

11. $y = \ln\left(\frac{5}{5-x}\right)$ Chain + Quotient

$y' = \frac{1}{\frac{5}{5-x}}\left(\frac{(5-x)(0) - 5(-1)}{(5-x)^2}\right)$

$y' = \frac{5-x}{5} \cdot \frac{+5}{(5-x)^2}$ $y' = \frac{1}{5-x}$

12. $y = \ln\sqrt{x^2 + 4}$ Chain but could've used prop. of logs $y = \frac{1}{2}\ln(x^2+4)$

$y = \ln(x^2+4)^{1/2}$

$y' = \frac{1}{\sqrt{x^2+4}} \cdot \frac{1}{2\sqrt{x^2+4}} \cdot 2x$

$y' = \frac{x}{x^2+4}$

13. $y = \ln(2 - \cos x)$ Chain

$y' = \frac{1}{2-\cos x} \cdot \sin x$

$y' = \frac{\sin x}{2 - \cos x}$

14. $y = \ln(5 - x)^6$ Rewrite with prop. of logs 1st

$y = 6\ln(5-x)$

$y' = 6 \cdot \frac{1}{5-x} \cdot -1$

$y' = \frac{-6}{5-x}$

15. $y = e^{\ln x^2}$ Rewrite

$y = \ln e \cdot x^2$

$y' = 2x$

16. $y = \ln(3x^2 + 2)^3$ Rewrite

$y = 3\ln(3x^2+2)$ chain

$y' = 3 \cdot \frac{1}{3x^2+2} \cdot 6x$

$y' = \frac{18x}{3x^2+2}$

17. $y = \ln x^3 + (\ln x)^3$ Rewrite / chain

$y = 3\ln x + (\ln x)^3$

$y' = 3 \cdot \frac{1}{x} + 3(\ln x)^2 \cdot \frac{1}{x}$

$y' = \frac{3 + 3(\ln x)^2}{x}$

or $\frac{3 + 3\ln^2 x}{x}$

18. $y = \ln\sqrt{\ln(x)}$ $\frac{d}{dx}\sqrt{\ln x} = \frac{1}{2\sqrt{\ln x}}$

$y' = \frac{1}{\sqrt{\ln x}} \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x}$

$y' = \frac{1}{2x\ln x}$

Limits and L'Hopitals Rule

Evaluate each Limit. Use L'Hopitals Rule when possible.

1. $\lim_{x\to 2} \frac{x^2-x-2}{x-2} = \frac{0}{0}$

$\frac{\frac{d}{dx}(x^2-x-2)}{\frac{d}{dx}(x-2)} = \frac{2x-1}{1}$ $\lim_{x\to 2} 2x-1$ $2(2)-1$ 3

2. $\lim_{x\to 2} \frac{x^3-8}{x^2-4} = \frac{0}{0}$

$\frac{\frac{d}{dx}x^3-8}{\frac{d}{dx}x^2-4} = \lim_{x\to 2} \frac{3x^2}{2x} = \frac{3(2)^2}{2(2)} = 3$

3. $\lim_{x\to 2} \frac{\sqrt{2+x}-2}{x-2}$

$\lim_{x\to 2} \frac{\frac{1}{2\sqrt{2+x}}}{1} = \frac{1}{2\sqrt{2+2}} = \frac{1}{4}$

4. $\lim_{x\to 0} \frac{\sin 5x}{x}$

$\lim_{x\to 0} \frac{\cos(5x)\cdot 5}{1} = \cos(0)\cdot 5 = 5$

5. $\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{2\cos(2x)}{3\cos(3x)} = \frac{2\cos(2\cdot 0)}{3\cos(3\cdot 0)} = \frac{2}{3}$

6. $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$ $\frac{x^{1/3}-1}{x-1}$

$\lim_{x\to 1} \frac{\frac{1}{3}x^{-2/3}}{1} = \lim_{x\to 1} \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{3}$

7. $\lim_{x\to \infty} \frac{x^2+2x+1}{x-1}$ $\frac{\infty}{\infty}$

$\lim_{x\to \infty} \frac{2x+2}{1} = \infty$

8. $\lim_{x\to 2} \frac{x^2-4x+4}{x^3-12x+16}$ $\frac{0}{0}$

$\lim_{x\to 2} \frac{2x-4}{3x^2-12} = \frac{0}{0}$

$\lim_{x\to 2} \frac{2}{6x} = \frac{2}{12} = \frac{1}{6}$

9. $\lim_{x\to \infty} \frac{x^2}{e^{5x}}$ $\frac{\infty}{\infty}$

$\lim_{x\to \infty} \frac{2x}{5e^{5x}}$ $\frac{\infty}{\infty}$

$\lim_{x\to \infty} \frac{2}{25e^{5x}} = \frac{2}{\infty} = 0$

10. $\lim_{x\to \infty} \frac{\ln x}{x}$

$\lim_{x\to \infty} \frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$

11. $\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{1+\cos 2x} = \frac{0}{0}$

$\lim_{x\to \pi/2} \frac{-\cos x}{-2\sin(2x)} = \frac{0}{0}$

$\lim_{x\to \pi/2} \frac{\sin x}{-4\cos(2x)} = \frac{1}{-4(-1)} = \frac{1}{4}$

12. $\lim_{x\to 3} \frac{2x-6}{x^2-9} = \frac{0}{0}$

$\lim_{x\to 3} \frac{2}{2x} = \frac{2}{6} = \frac{1}{3}$

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Derivatives Review

1. Find the average rate of change of $f(x)=5\sqrt{x^2+9}-2$ over [0, 4].

$f(0)=13$

$f(4)=23$ $\quad m=\frac{23-13}{4-0}=\frac{10}{4}=\frac{5}{2}$

2. Find the Equation of the tangent line: $f(x)=4x^2-5x+2$ at $x=3$

$f(3)=4(3)^2-5(3)+2=23 \quad (3,23)$

$f'(x)=8x-5$

$f'(3)=8(3)-5 \quad y-23=19(x-3)$

$m=19$

3. Find the Equation of the normal line: $f(x)=4x^2-5x+2$ at $x=3$

$(3,23) \quad m=19 \quad \perp m=-\frac{1}{19}$

$y-23=-\frac{1}{19}(x-3)$

4. Find the Derivative using the definition: $f(x)=\sqrt{2x+3}$

$f'(x)=\lim_{h\to 0}\frac{\sqrt{2(x+h)+3}-\sqrt{2x+3}}{h}\cdot\frac{(\sqrt{2(x+h)+3}+\sqrt{2x+3})}{(\sqrt{2(x+h)+3}+\sqrt{2x+3})}$

$\lim_{h\to 0}\frac{2(x+h)+3-(2x+3)}{h(\sqrt{2(x+h)+3}+\sqrt{2x+3})}=\frac{\cancel{2x}+2h\cancel{+3}\cancel{-2x}\cancel{-3}}{h(\sqrt{2(x+h)+3}+\sqrt{2x+3})}$

$\lim_{h\to 0}\frac{2}{\sqrt{2(x+0)+3}+\sqrt{2x+3}}=\frac{2}{2\sqrt{2x+3}} \quad f'(x)=\frac{1}{\sqrt{2x+3}}$

5. Find the Derivative using the definition: $f(x)=\frac{3}{2x-1}$

$f'(x)=\lim_{h\to 0}\frac{\frac{3}{2(x+h)-1}-\frac{3}{2x-1}}{h}=$

$\frac{\frac{3(2x-1)-3(2(x+h)-1)}{(2(x+h)-1)(2x-1)}}{h}=\frac{6x-3-6x-6h+3}{(2(x+h)-1)(2x-1)}\cdot\frac{1}{h}$

$\lim_{h\to 0}\frac{-6}{(2(x+0)-1)(2x-1)}=\frac{-6}{(2x-1)^2}$

6. Find the Derivative using the definition: $f(x)=2x^2+3x-5$

$f'(x)=\lim_{h\to 0}\frac{2\overbrace{(x+h)^2}^{(x^2+2xh+h^2)}+3(x+h)-5-(2x^2+3x-5)}{h}$

$\lim_{h\to 0}\frac{\cancel{2x^2}+4xh+2h^2\cancel{+3x}+3h\cancel{-5}\cancel{-2x^2}\cancel{-3x}\cancel{+5}}{h}$

$\lim_{h\to 0}\frac{\cancel{h}(4x+2h+3)}{\cancel{h}}=4x+2(0)+3 \quad f'(x)=4x+3$

7. Find the Derivative using the definition: $f(x)=2-\frac{3}{x}$

$f'(x)=\lim_{h\to 0}\frac{\cancel{2}-\frac{3}{x+h}\cancel{-2}+\frac{3}{x}}{h}=\lim_{h\to 0}\frac{\frac{-3x}{x(x+h)}+\frac{3x+3h}{x(x+h)}}{h}$

$f'(x)=\lim_{h\to 0}\frac{3\cancel{h}}{x(x+h)}\cdot\frac{1}{\cancel{h}}=\frac{3}{x(x+0)}$

$f'(x)=\frac{3}{x^2}$

Find the Derivative.

8. $f(x)=\frac{x^2+3x+2}{x^2-1}$

$f'(x)=\frac{(x^2-1)(2x+3)-(x^2+3x+2)(2x)}{(x^2-1)^2}$

$=\frac{2x^3+3x^2-2x-3-2x^3-6x^2-4x}{(x^2-1)^2} \quad f'(x)=\frac{-3x^2-6x-3}{(x^2-1)^2}$

9. $f(x)=\frac{x+1}{\sqrt{x}}=\frac{x+1}{x^{1/2}}=f(x)=x^{1/2}+x^{-1/2}$

$f'(x)=\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}$

$f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x^3}}$ or $\frac{1}{2x^{1/2}}-\frac{1}{2x^{3/2}}$

10. $f(x)=\sin^2 3x$

$f(x)=(\sin 3x)^2$

$f'(x)=2(\sin(3x))(\cos(3x))\cdot 3$

$f'(x)=6\sin(3x)\cos(3x)$

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11. $f(x)=\frac{x(x^2-1)}{x+1}$ or $f(x)=\frac{x^3-x}{x+1}$

$f(x)=\frac{x\cancel{(x+1)}(x-1)}{\cancel{x+1}}$

$f(x)=x^2-x$

$f'(x)=2x-1$

or

$f'(x)=\frac{(x+1)(3x^2-1)-(x^3-x)(1)}{(x+1)^2}$

$=\frac{3x^3+3x^2-x-1-x^3+x}{(x+1)^2}$

$f'(x)=\frac{2x^3+3x^2-1}{(x+1)^2}$

12. $y = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$

$y = (x^4 - x^3 + x^2 - x)(x^2 + x + 1)$

$y = x^6 + \cancel{x^5} + \cancel{x^4} - \cancel{x^5} - \cancel{x^4} - \cancel{x^3} + x^4 + \cancel{x^3} + \cancel{x^2} - x^3 - \cancel{x^2} - x$

$y = x^6 + x^4 - x^3 - x$

$y' = 6x^5 + 4x^3 - 3x^2 - 1$

13. Product

$y = x \cdot \sin x + \cos x$

$y' = x\cos x + \sin x(1) + -\sin x$

$y' = x\cos x$

14. $y = 5\sec x + \tan x$

$y' = 5\sec x\tan x + \sec^2 x$

15. $f(x) = \sqrt{x} + 4\csc x =$

$f'(x) = \frac{1}{2\sqrt{x}} + 4(-\csc x\cot x)$

$f'(x) = \frac{1}{2\sqrt{x}} - 4\csc x\cot x$

16. $f(x) = \frac{1}{x} - 10\sec x = x^{-1} - 10\sec x$

$f'(x) = \frac{-1}{x^2} - 10\sec x\tan x$

$f'(x) = \frac{-1}{x^2} - 10\sec x\tan x$

17. $y = \left(\frac{x+1}{x+2}\right)\frac{(2x-5)}{1} = \frac{2x^2 - 3x - 5}{x+2}$

$y' = \frac{(x+2)(4x-3) - (2x^2 - 3x - 5)(1)}{(x+2)^2}$

$y' = \frac{4x^2 + 5x - 6 - 2x^2 + 3x + 5}{(x+2)^2}$

$y' = \frac{2x^2 + 8x - 1}{(x+2)^2}$

18. $y = (2x - 7)^3$

$y' = 3(2x-7)^2(2)$

$y' = 6(2x-7)^2$

19. $f(x) = (9 - x^2)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}(9-x^2)^{-1/3}(-2x)$

$f'(x) = \frac{-4x}{3\sqrt[3]{9-x^2}}$ or $f'(x) = \frac{-4x}{3(9-x^2)^{1/3}}$

20. $y = \sqrt{\frac{1}{4x^2}}$ $y = \frac{1}{2x} = (2x)^{-1}$

$y' = -1(2x)^{-2}(2)$

$y' = \frac{-2}{(2x)^2} = \frac{-2}{4x^2}$ $y' = \frac{-1}{2x^2}$

21. $f(x) = \frac{x^2}{x^2 + 3}$

$f'(x) = \frac{(x^2+3)(2x) - x^2(2x)}{(x^2+3)^2} = \frac{2x^3 + 6x - 2x^3}{(x^2+3)^2}$

$f'(x) = \frac{6x}{(x^2+3)^2}$

22. $y = \frac{\cos \pi x + 1}{x}$

$y' = \frac{x(-\sin(\pi x)(\pi) + 0) - (\cos(\pi x) + 1)(1)}{x^2}$

$y' = \frac{-\pi x\sin(\pi x) - \cos(\pi x) - 1}{x^2}$

23. $f(x) = 3\tan 4x$

$f'(x) = 3\sec^2(4x)4$

$f'(x) = 12\sec^2(4x)$

24. $y = \sin(\cos x)$

$y' = \cos(\cos x)(-\sin x)$

$y' = -\sin x\cos(\cos x)$

25. $y = 3x - 5\cos(\pi x)^2$ double chain

$y' = 3 - 10\cos(\pi x)(-\sin(\pi x))(\pi)$

$y' = 3 + 10\pi\cos(\pi x)\sin(\pi x)$

26. $f(x) = \sqrt{x^2 + 2x + 8}$

$f(x) = (x^2 + 2x + 8)^{1/2}$

$f'(x) = \frac{1}{2}(x^2+2x+8)^{-1/2}(2x+2)$

$f'(x) = \frac{2x+2}{2\sqrt{x^2+2x+8}}$ or $\frac{2(x+1)}{2\sqrt{x^2+2x+8}}$

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$f'(x) = \frac{x+1}{\sqrt{x^2+2x+8}}$

27. $y = \frac{1}{x} + \sqrt{\cos x}$ $y = x^{-1} + (\cos x)^{1/2}$

$y' = \frac{-1}{x^2} + \frac{1}{2}(\cos x)^{-1/2} \cdot -\sin x$

$y' = \frac{-1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$

Product

28. $y = (x^2 + 1)e^{3x}$

$y' = (x^2+1)e^{3x}\cdot 3 + e^{3x}(2x)$

$y' = 3e^{3x}(x^2+1) + 2xe^{3x}$

29. $y = x \cdot 5^{3x}$

$y' = x\cdot 5^{3x}\ln 5\cdot 3 + 5^{3x}\cdot 1$

$y' = 3x\ln 5\cdot 5^{3x} + 5^{3x}$

or $y' = 5^{3x}(3x\ln 5 + 1)$

30. $y = \sin(e^{2x})$

$y' = \cos(e^{2x})e^{2x}\ln e\cdot 2$

$y' = 2e^{2x}\cos(e^{2x})$

31. $y = e^{e^{5x}}$

$y' = e^{e^{5x}}\cdot\cancel{\ln e}\cdot e^{5x}\cdot\cancel{\ln e}\cdot 5$

$y' = 5e^{e^{5x}}e^{5x}$

32. $y = 3^{x^2+3x}$

$y' = 3^{x^2+3x}\cdot\ln 3\cdot(2x+3)$

33. $y = (\ln x)^x$ Prop of Logs

$y = x\ln x$

$y = x\cdot\frac{1}{x} + \ln x\cdot 1$

$y = 1 + \ln x$

34. $y = 3^{5x}$

$y' = 3^{5x}\ln 3\cdot 5$

$y' = 5\ln(3)(3^{5x})$

35. $y = e^{\ln(5x^2)}$ Prop

$y = \cancel{\ln e}\,5x^2$

$y' = 10x$

36. Find the first TWO Derivatives:

$y = 2(x^2 - 1)^3$

$y' = 6(x^2-1)^2(2x)$

$y' = 12x(x^2-1)^2$

$y'' = 12x\cdot 2(x^2-1)(2x) + (x^2-1)^2\cdot 12$

$y'' = 48x^2(x^2-1) + 12(x^2-1)^2$ or $60x^4 - 72x^2 + 12$

37. Find the f', f'', and f'''

$f(x) = x^3 + 2x^2 - 4x + 5$

$f'(x) = 3x^2 + 4x - 4$

$f''(x) = 6x + 4$

$f'''(x) = 6$

Evaluate the Limit using L'Hopital's Rule:

38. $\lim_{x\to 3}\frac{x^2 + 4x - 21}{x^2 - 7x + 12} = \frac{0}{0}$

$\lim_{x\to 3}\frac{2x+4}{2x-7}$

$\frac{2(3)+4}{2(3)-7} = \frac{10}{-1} = -10$

39. $\lim_{x\to 0}\frac{\tan 3x}{\ln(1+x)} = \frac{0}{0}$

cos $(1,0)$

$\lim_{x\to 0}\frac{\sec^2(3x)(3)}{\frac{1}{1+x}} = \frac{3\sec^2(0)}{\frac{1}{1+0}} = \frac{3(1)}{1} = 3$

40. $\lim_{x\to 1}\frac{x^2 - 1}{x - 1}$ $\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$

$\lim_{x\to 1}\frac{2x}{1} = 2(1) = 2$

41. $\lim_{x\to 3}\frac{\sqrt{x+6} - 3}{x - 3} = \frac{0}{0}$

$\lim_{x\to 3}\frac{(x+6)^{1/2} - 3}{x-3} = \frac{\frac{1}{2\sqrt{x+6}}}{1} = \frac{1}{2\sqrt{3+6}} = \frac{1}{2\cdot 3} = \frac{1}{6}$

$2\sqrt{9}$

42. $\lim_{x\to 0}\frac{x}{\frac{1}{3+x} - \frac{1}{3}} = \frac{0}{0}$ $\frac{x}{(3+x)^{-1} - \frac{1}{3}}$

$\lim_{x\to 0}\frac{1}{-1(3+x)^{-2} + 0}$

$\lim_{x\to 0}\frac{-1}{(3+x)^{-2}} = -1(3+x)^2 = -1(3+0)^2$

-9

17

43. $\lim_{x\to\infty}\frac{x - 4}{x^2 - 6x + 8}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty}\frac{1}{2x-6} = \frac{1}{\infty} = 0$

Unit 3 Review – Derivatives

1. Use the long definition of derivative to find $f'(x)$. Then find the slope, and the equation of the tangent line to the curve at the given value. Graph both the function and the tangent on the graph.

   $f(x) = x^2 - 1$ at $x = 2$ $\quad f(2)=3$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-1-(x^2-1)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2\cancel{-1}\cancel{-x^2}\cancel{+1}}{h}$$

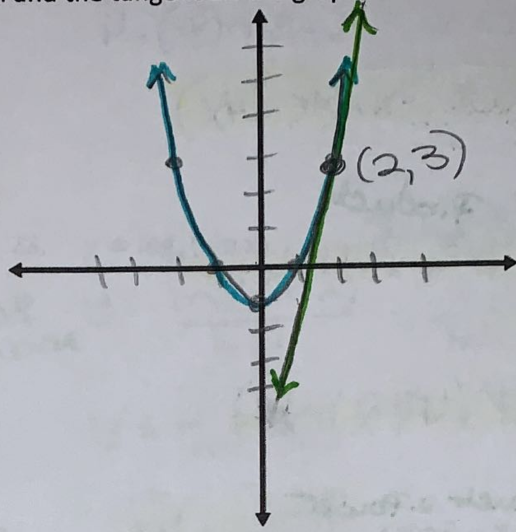
$$f'(x)=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$f'(x)=2x$ $\quad f'(2)=4$

$m=4$

$y-3=4(x-2)$

or $y=4x-5$



Given the following functions, find:

2. $f'''(-1)$ where $f(x) = \frac{5}{x^4}$

$f(x)=5x^{-4}$

$f'(x)=-\frac{20}{x^5}=-20x^{-5}$

$f''(x)=\frac{100}{x^6}=100x^{-6}$

$f'''(x)=-\frac{600}{x^7}$

$f'''(-1)=\frac{-600}{(-1)^7}=600$

3. $f''(-2)$ where $f(x) = \frac{1}{x+1}$

$f(x)=(x+1)^{-1}$

$f'(x)=\frac{-1}{(x+1)^2}=-1(x+1)^{-2}$

$f''(x)=\frac{2}{(x+1)^3}$

$f''(-2)=\frac{2}{(-2+1)^3}=\frac{2}{(-1)^3}=-2$

Find the following derivatives using power, product, quotient, or chain rule.

4. $y = \sqrt{x} + \frac{1}{x}$ $= x^{1/2}+x^{-1}$

$y'=\frac{1}{2}x^{-1/2}-1x^{-2}$

$y'=\frac{1}{2\sqrt{x}}-\frac{1}{x^2}$

5. $y = \tan(3x^2 - 5)$

$y'=\sec^2(3x^2-5)\,(6x)$

$y'=6x\sec^2(3x^2-5)$

6. $y = \frac{1}{\sqrt[3]{3-x^3}}$ $=(3-x^3)^{-1/3}$

$y'=-\frac{1}{3}(3-x^3)^{-4/3}(-3x^2)$

$y'=\frac{\cancel{3}x^2}{\cancel{3}(3-x^3)^{4/3}}$

$y'=\frac{x^2}{(3-x^3)^{4/3}}$

7. $y = x^2 \sin x$

$y'=x^2\cos x+\sin x\cdot 2x$

$y'=x^2\cos x+2x\sin x$

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8. $y=\sqrt{\cos^3(4x)} = (\cos(4x)^3)^{1/2}$ Rewrite

$y=(\cos(4x))^{3/2}$

Double Chain $y' = \frac{3}{\cancel{2}}(\cos(4x))^{1/2}\cdot -\sin(4x)\cdot \cancel{4}^{2}$

$y' = -6\sqrt{\cos(4x)}\sin(4x)$

9. $y=\sin^2(4x) = (\sin(4x))^2$ Double Chain

$y' = 2(\sin(4x))\cos(4x)\cdot 4$

$y' = 8\sin(4x)\cos(4x)$

10. $y=4^x x^4$ Product

$y' = 4^x\cdot 4x^3 + x^4\cdot 4^x\ln 4$

or

$y' = x^3 4^x(4+x\ln 4)$

11. $y=x(3x-9)^4$ Product + Chain

$y' = x\cdot 4(3x-9)^3\cdot 3 + (3x-9)^4\cdot 1$

$y' = 12x(3x-9)^3 + (3x-9)^4$

12. $y=x^3\sin x - 5\cos x$ Product + Power

$y' = x^3\cdot\cos x + \sin x\cdot 3x^2 - 5(-\sin x)$

$y' = x^3\cos x + 3x^2\sin x + 5\sin x$

13. $y=x^3e^x$ Product

$y' = x^3\cdot e^x + e^x\cdot 3x^2$

or

$y' = x^2e^x(x+3)$

14. $y=\frac{\ln x}{x^2}$ Quotient

$y' = \frac{x^2\cdot\frac{1}{x} - \ln x\cdot 2x}{(x^2)^2}$

$y' = \frac{x-2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4}$

or $y' = \frac{1-2\ln x}{x^3}$

15. $y=\frac{\sin x}{x}$ Quotient

$y' = \frac{x\cdot\cos x - \sin x\cdot 1}{x^2}$

$y' = \frac{x\cos x - \sin x}{x^2}$

16. $y=e^{-x^3}$

$y' = e^{-x^3}\cancel{\ln e}\cdot -3x^2$

$y' = \frac{-3x^2}{e^{x^3}}$

17. $y=7^{\ln x}$

$y' = 7^{\ln x}\ln 7\cdot\frac{1}{x}$

$y' = \frac{7^{\ln x}\ln 7}{x}$

18. $y=e^x - xe^x$ product

$y' = e^x-(xe^x+e^x)$

$y' = e^x - xe^x + e^x$

$y' = -xe^x$

19. $y=\frac{e^x}{x^2-1}$ Quotient

$y' = \frac{e^x(x^2-1)-e^x2x}{(x^2-1)^2}$

$y' = \frac{(x^2-1)e^x - e^x(2x)}{(x^2-1)^2} = \frac{e^x(x^2-1-2x)}{(x^2-1)^2}$

or

$y' = \frac{e^x(x^2-2x-1)}{(x^2-1)^2}$

20. $y = \ln\sqrt{x^2-4}$ Rewrite $= \ln(x^2-4)^{1/2}$

Prop of Logs $y = \frac{1}{2}\ln(x^2-4)$

$y' = \frac{1}{\cancel{2}} \cdot \frac{1}{x^2-4} \cdot \cancel{2}x$

$y' = \frac{x}{x^2-4}$

21. $y = 7^{e^x}$

$y' = 7^{e^x} \ln 7 \cdot e^x$

22. $y = \log_3 x^2$ Rewrite $\frac{\ln \#}{\ln \text{base}}$

$y = \frac{\ln x^2}{\ln 3}$ (or you could rewrite as a product like #23)

Quotient

$y' = \frac{\ln 3 \cdot \frac{1}{x^2} \cdot 2x - \ln x^2 (0)}{(\ln 3)^2}$

$\frac{2\cancel{\ln 3}}{x} \cdot \frac{1}{(\ln 3)^2}$ $y' = \frac{2}{x\ln 3}$

23. $y = \log_4(\sin x)$ Rewrite as a product

$\frac{\ln \#}{\ln \text{base}}$ $y = \frac{\ln(\sin x)}{\ln 4} = \underbrace{\frac{1}{\ln 4}}_{1st} \cdot \underbrace{\ln(\sin x)}_{2nd}$

$y' = \frac{1}{\ln 4} \cdot \frac{1}{\sin x} \cdot \cos x + \ln(\sin x)(0)$

$y' = \frac{\cot x}{\ln 4}$

24. $y = 4^x \sin x$

$y' = 4^x \cdot \cos x + \sin x \cdot 4^x \ln 4$

or

$y' = 4^x(\cos x + \ln 4 \sin x)$