3.1 The Average Rate of Change + The Definition of a Derivative at a Point

Slope of the Secant Line

$m_{sec} = \dfrac{f(x) - f(a)}{x - a}$ This formula represents the AVERAGE rate of change.

Find the average rate of change of $f(x) = x^2 - 5x$ from $[2, 5]$

$$m_{sec} = \frac{f(5) - f(2)}{5 - 2}$$

$$= \frac{(25 - 25) - (4 - 10)}{5 - 2}$$

$$= \frac{0 + 6}{3}$$

$$m_{sec} = 2$$

Derivative:

- Slope of a tangent line
- Rate of change at $x = a$
- Instantaneous rate of change or derivative
- Denoted by y', $f'(x)$, or dy/dx

Def. of a Derivative at a Point

$(a, f(a))$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*Use this version when you are given a specific x-value to evaluate the derivative at.

Slope of the Tangent Line

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

This formula represents the INSTANTANEOUS rate of change (IROC)

Find the derivative at a given point.

1. $f(x) = 2x + 3$ at $x = 1$

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(1) = \lim_{x \to 1} \frac{(2x + 3) - (2(1) + 3)}{x - 1}$$

$$f'(1) = \lim_{x \to 1} \frac{2x + 3 - 5}{x - 1}$$

$$f'(1) = \lim_{x \to 1} \frac{2(x - 1)}{x - 1}$$

$$f'(1) = \lim_{x \to 1} 2$$

$$f'(1) = 2$$

2. $f(x) = \frac{2x+1}{x+2}$ at $(\overset{a}{1}, \overset{f(a)}{1})$

$f'(1) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$f'(1) = \lim_{x \to 1} \frac{\frac{2x+1}{x+2} - 1}{x - 1}$

$f'(1) = \lim_{x \to 1} \frac{\frac{2x+1-x-2}{x+2}}{x-1}$

$f'(1) = \lim_{x \to 1} \left(\frac{x-1}{x+2} \cdot \frac{1}{x-1}\right)$

$f'(1) = \lim_{x \to 1} \frac{1}{x+2}$

$f'(1) = \frac{1}{1+2}$

$f'(1) = \frac{1}{3}$

Tangent + Normal Equations

To find the tangent equation:
- find the slope of the tangent line (find the derivative)
- input the slope + the point into point-slope form of a line

To find the equation of the normal line:
- find the slope of the tangent line (derivative)
- find the slope of the line perpendicular to the tangent line (opposite reciprocal)
- input the slope & the point into point-slope form of a line

Find the derivative of the function at the given point. Then find the equation of the tangent + normal line.

$f(x) = x^2 + 2x$ find the $f'(3)$.

$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$f'(3) = \lim_{x \to 3} \frac{(x^2+2x) - f(3)}{x-3}$

$f'(3) = \lim_{x \to 3} \frac{x^2+2x-15}{x-3}$

$f'(3) = \lim_{x \to 3} \frac{(x+5)(x-3)}{x-3}$

$f'(3) = \lim_{x \to 3} x+5$

$f'(3) = 8$

$f(3) = (3)^2 + 2(3) = 15$

★Point-Slope form of a line
$y - y_1 = m(x - x_1)$

$f(3) = 15 \quad m = 8$

Slope of the tangent line:
$y - 15 = 8(x-3)$

Slope of the normal line:
$y - 15 = -\frac{1}{8}(x-3)$

Velocity

Instantaneous velocity: $v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

This means that the velocity at a time $t=a$ is equal to the slope of the tangent line.

If a ball is thrown into the air with a velocity of 40 ft/sec, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t=2$.

$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$v(2) = \lim_{h \to 0} \frac{(40(2+h) - 16(2+h)^2) - f(2)}{h}$

$v(2) = \lim_{h \to 0} \frac{40(2+h) - 16(2+h)^2 - (40(2) - 16(2)^2)}{h}$

$v(2) = \lim_{h \to 0} \frac{80 + 40h - 16(4 + 4h + h^2) - 16}{h}$

$v(2) = \lim_{h \to 0} \frac{80 + 40h - 64 - 64h - 16h^2 - 16}{h}$

$v(2) = \lim_{h \to 0} \frac{-16h^2 - 24h}{h}$

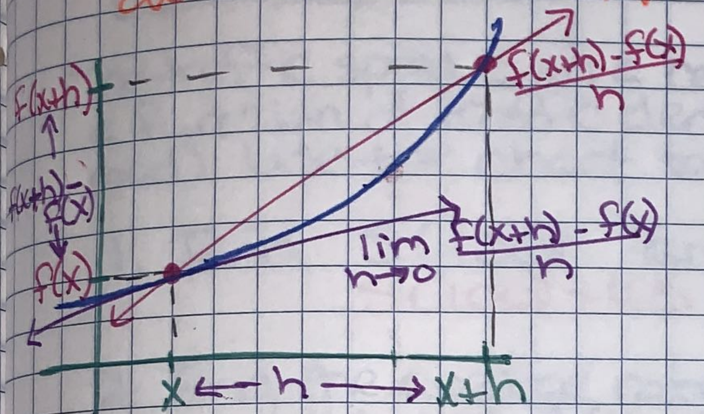
$v(2) = \lim_{h \to 0} \frac{h(-16h - 24)}{h}$

$v(2) = -16(0) - 24$

$v(2) = -24$ ft/sec

Definition of a Derivative (Formal definition)

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$



Label the following: $f(x)$, $f(x+h)$, h, $f(x+h)-f(x)$, + a segment whose slope represents $\frac{f(x+h)-f(x)}{h}$ + $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Examples:

1. Find the equation of the tangent line to $f(x) = -2x^2+3x-4$ at $x=2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-2(x+h)^2+3(x+h)-4-(-2x^2+3x-4)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-2x^2-4xh-2h^2+3x+3h-4+2x^2-3x+4}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-4xh-2h^2+3h}{h}$$

$$f'(x) = \lim_{h\to 0} -4x-2h+3$$

$$f'(x) = -4x-2(0)+3$$

$$f'(2) = -4(2)+3$$

$$f'(2) = -5$$

Equation of tangent line

$f(2) = -2(2)^2+3(2)-4$

$f(2) = -6$

$$y+6 = -5(x-2)$$

2. Find the derivative of $f(x) = \frac{3}{x-2}$

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{3}{x+h-2} - \frac{3}{x-2}\right)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{3(x-2)-3(x+h-2)}{(x+h-2)(x-2)}\right)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{3x-6-3x-3h+6}{(x+h-2)(x-2)}\right)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-3\cancel{h}}{(x+h-2)(x-2)} \cdot \frac{1}{\cancel{h}}$$

$$f'(x) = \lim_{h\to 0} \frac{-3}{(x+h-2)(x-2)}$$

$$f'(x) = \frac{-3}{(x+0-2)(x-2)}$$

$$f'(x) = \frac{-3}{(x-2)^2}$$

Derivatives from a Chart

q (density)	100	110	120	130	140
S (speed)	45	42	39.5	37	35

The traffic speed S on a certain road (in mph) varies as a function of traffic density q (# of cars per mile on the road). Use the chart to answer the following questions

1. Find $S(100)$ and $S(120)$

   $S(100) = 45$ $S(120) = 39.5$

2. Find the average rate of change from $q = 110$ to $q = 120$

   $(110, 42)$ $(120, 39.5)$
   x_1 y_1 x_2 y_2

   $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{39.5 - 42}{120 - 110} = \frac{-2.5}{10}$

   $m_{sec} = -0.25$ mph / car per mile

3. Estimate the instantaneous rate of change at $q = 130$

   $m_{sec\,(130\,to\,140)} = \frac{35 - 37}{140 - 130} = \frac{-2}{10} = -0.2$

   $m_{sec\,(120\,to\,130)} = \frac{37 - 39.5}{130 - 120} = \frac{-2.5}{10} = -0.25$

   $m_{tan} \approx \text{avg}(m_{sec(130-140)} \text{ and } m_{sec(120-130)})$

   $m_{tan} \approx \frac{-0.2 + -0.25}{2}$

   $m_{tan} \approx -0.225$

4. Estimate the instantaneous rate of change at $q = 140$

   $m_{sec\,(130\,to\,140)} = \frac{35 - 37}{140 - 130} = \frac{-2}{10}$

   $m_{tan} \approx m_{sec\,(130\,to\,140)}$

   $m_{tan} \approx -0.2$

Derivatives from a Chart

q (density)	100	110	120	130	140
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1. Find S(100) and S(120)

$S(100) = 45$ $\quad$ $S(120) = 39.5$

2. Find the average rate of change from $q = 110$ to $q = 120$

$(110, 42)$ $\quad$ $(120, 39.5)$
$x_1 \; y_1 \quad x_2 \; y_2$

$$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{39.5 - 42}{120 - 110} = \frac{-2.5}{10}$$

$m_{sec} = -0.25$ mph / car per mile

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$m_{tan} \approx m_{sec\,(130\,to\,140)}$

$m_{tan} \approx -0.2$

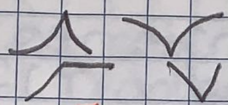
Cont. 3.2

p.39

when is a graph differentiable?

A graph is not differentiable anywhere the following is true:

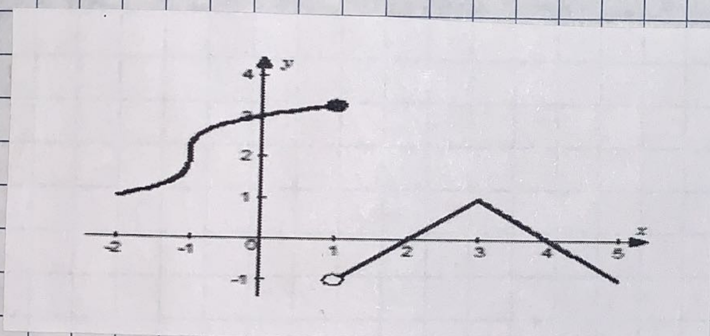
- Cusp
- Corner



- Vertical Tangent →

- Discontinuity
  - Removable
  - Infinite
  - Jump

State the x values where f is not differentiable and the reason



X = -1 Vertical Tangent
X = 1 Jump Discontinuity
X = 3 Corner

3.3 - Interpreting the Derivative

Script: The instantaneous rate of change in "y" (in context) when "x" (in context with value) is "derivative" (in context with units)

1. The cost of extracting T tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say that $f'(2000) = 100$?
   Step 1: Identify Variables
   x = tons of ore = 2000 tons
   y = cost of extracting ore
   Step 2: Fill in the script
   The IROC in the cost of extracting 2000 T of ore is 100 dollars per ton.

2. Suppose $P = f(t)$ is the population of Mexico in millions, where t is the # of years since 1980. Explain the meanings of the statements:
   x = # of yrs since 1980
   y = population of mexico in millions of people
   - $f(5) = 15$ In 1985, the population of Mexico was 15 million people.
   - $f'(6) = 2$ The IROC in the population in Mexico in 1986 was 2 million people per year.
   - $f^{-1}(95.5) = 16$ Mexico had a population of 95.5 million people in 1996.

   *** The order in which you say this matters. Discuss x 1st!

   *** Discuss y 1st!

3. A rod of length 12 cm is heated at one end. The table below gives the temperature T(x) in degrees Celsius at selected number x cm from the heated end.

x	0	2	5	7	9	12
T(x)	80	71	66	60	54	50

- Use the table to approximate $T'(8)$

$$T'(8) \approx \frac{54-60}{9-7}$$

$$T'(8) \approx -3\,°C/cm$$

- Using appropriate units, interpret $T'(8)$ in the context of the problem
  Identify variables: x = # of cm from heated end of the rod
  y = temperature in degrees Celsius
  The IROC in temperature when the rod is measured 8 cm from the heated end is decreasing approximately 3°C/cm.

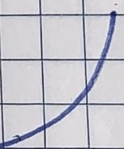
3.4 Curve Sketching f to f'

RELATIONSHIP BETWEEN f, f', f''

f	f'	f''
-Cusp -Corner -Discontinuity -Removable -Infinite -jump -Vertical Tangent	DNE	DNE
Local max, local min (local extrema), horizontal tangent	0 On the x-axis	
f increasing	Positive (Above the x-axis)	
f decreasing	Negative (Below the x-axis)	
f concave up	Increasing	Positive (Above the x-axis)
f concave down	Decreasing	Negative (Below the x-axis)
Points of Inflections	Local Extrema	Change Signs

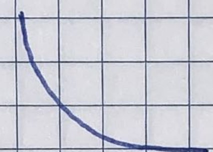
What can we say about g, g', g" for each segment of the graph y = g(x)?

1.



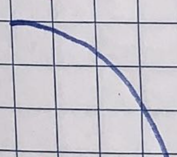
g is increasing + concave up
g' is positive + increasing
g" is positive

2.



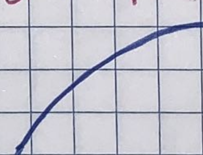
g is decreasing + concave ↑
g' is negative + increasing
g" is positive

3.



g is decreasing + concave down
g' is negative + decreasing
g" is negative

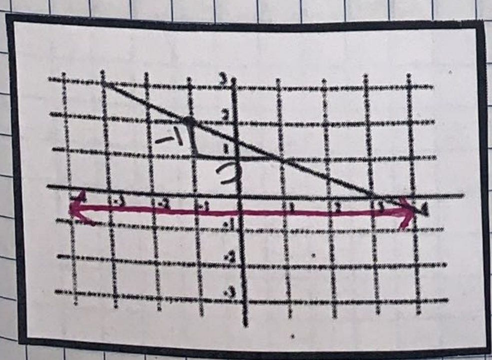
4.



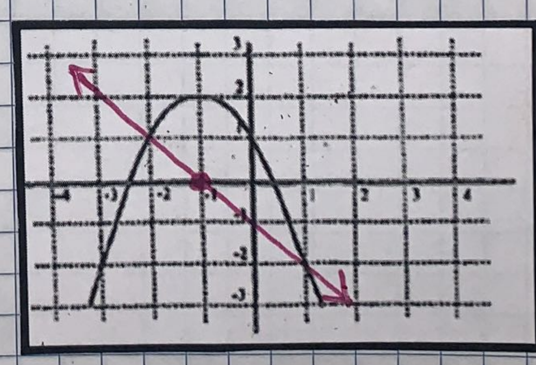
g is increasing + concave ↓
g' is positive + decreasing
g" is negative

Graph the First Derivatives

1.



2.

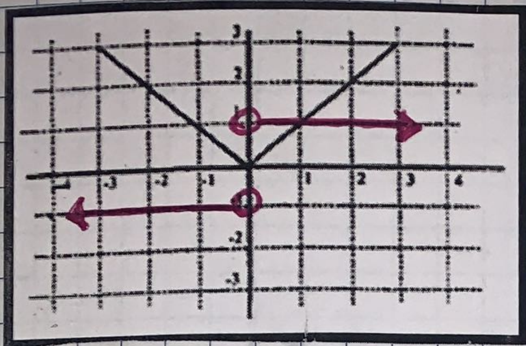


f' -1/2 -1/2 -1/2

f' + + + 0 - - -
above -1 below

cont. 3.4

3.



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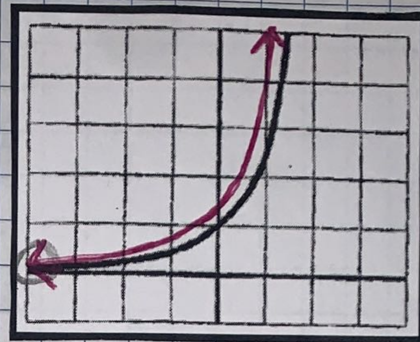


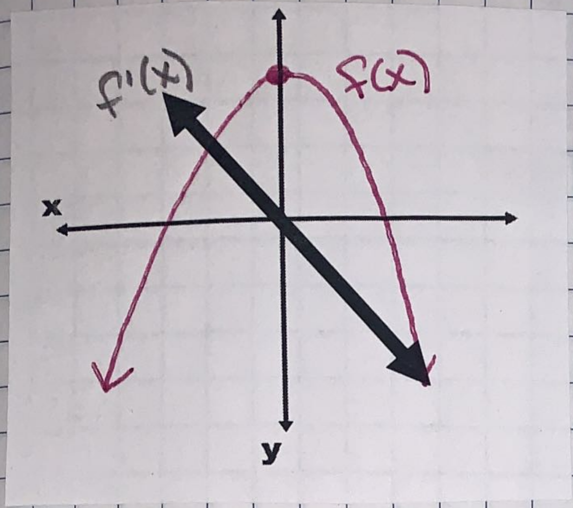
f'(x) -1 -1 -1 DNE 1 1 1
0

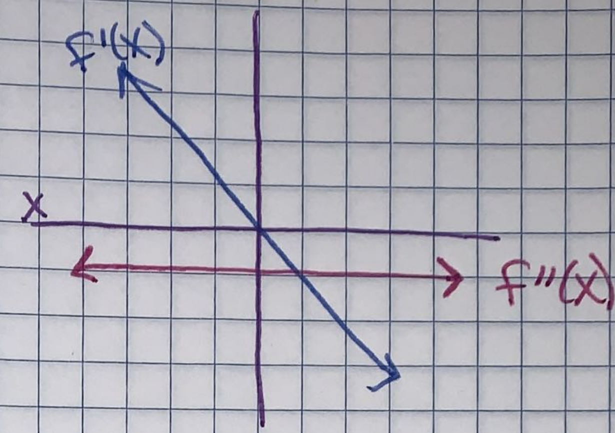
f'(x) close to 0 + + + + +

3.5 Curve Sketching f from f'

From f', sketch f and f'':

1.





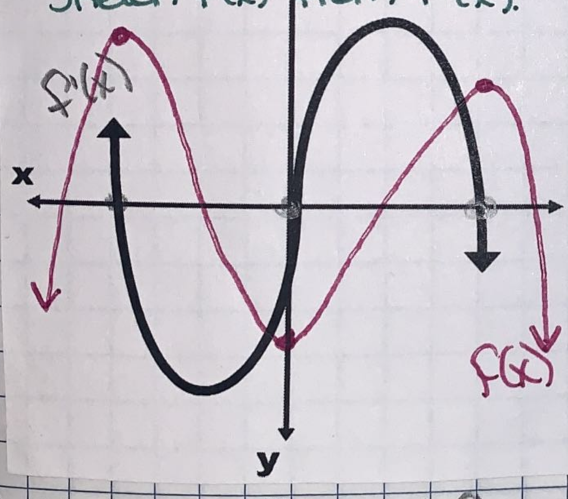
f'(x) + + + + | − − − −
0

f(x) Incr ↗ max Decr ↘

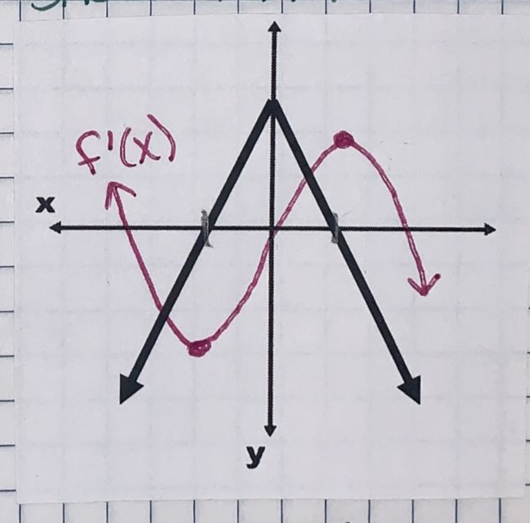
f''(x) −1 −1 −1 −1

Sketch f(x) from f'(x)

2. Sketch f(x) from f'(x).



3.



f'(x) + + 0 − − − − 0 + + + + 0 − −

f(x) Inc ↗ max Dec ↘ min Incr ↗ max Decr ↘

f'(x) − − 0 + + + 0 − − −

f(x) Decr ↘ min Incr ↗ max Decr ↘

Draw a possible graph of f(x) given the info below:

a. f(x) is continuous
b. $f(3) = 2$
c. $f'(x) > 0$, $(-\infty, 0) \cup (3, \infty)$
d. $f'(x) < 0$, $(0, 3)$
e. $f'(x) = 0$ at $x = 0$, $x = 3$

f'(x) + + | − − | + +
0 3

f(x) Incr ↗ | Decr ↘ | Incr ↗
0 max 3 min