2.1 Limits from Graphs and Graphs from Limits

Definition of a Limit

f(x) gets close to some limit as x gets close to a value.

f(x) gets close to L as x gets close to a.

$f(x) \to L$ as $x \to a$

$$\lim_{x \to a} f(x) = L$$

Examples - Explain the meaning of:

1. $\lim_{x \to 2} f(x) = 5$ — f(x) approaches 5 as x approaches 2 (from the left + right of 2)
2. $\lim_{x \to -2^+} f(x) = 3$ — f(x) approaches 3 as x approaches −2 from the right
3. $\lim_{x \to \infty} f(x) = 0$ — f(x) approaches 0 as x approaches positive infinity
4. $\lim_{x \to 1^-} f(x) = \infty$ — f(x) approaches positive infinity as x approaches 1 from the left

Limits from Tables - Find the following limits:

5.

x	8.9	8.99	8.999	8.9999	9	9.001	9.01	9.1
f(x)	5.98329	5.99883	5.99983	5.999983	6	6.00016	6.00166	6.016
g(x)	15.21	15.9201	15.99201	15.999200	und	16.00080	16.0801	16.81
h(x)	5.98329	5.99883	5.99983	5.999983	6	16.00080	16.0801	16.81

a) $\lim_{x \to 9} f(x) = 6$

It doesn't matter what f(9) equals. It's what f(x) is approaching from left + right

b) $\lim_{x \to 9} f(x) = 16$

g(9) doesn't have to exist

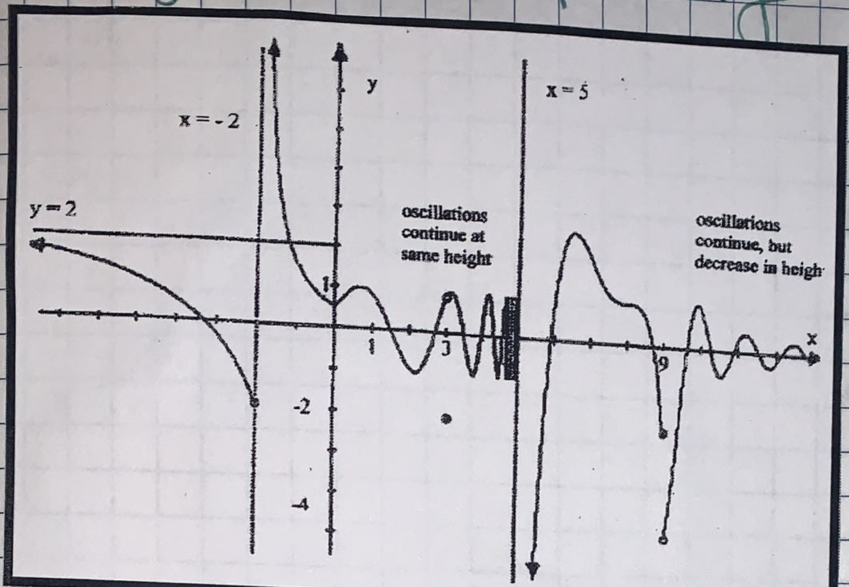
c) $\lim_{x \to 9} h(x)$ DNE

* no equal sign for DNE

cont. 2.1

Finding Limits Graphically

6.



a. $\lim_{x \to -\infty} g(x) = 2$

e. $\lim_{x \to 0} g(x) \approx .75$

i. $\lim_{x \to \infty} g(x) = 0$

b. $\lim_{x \to 9^-} g(x) = -2$

f. $\lim_{x \to 5^-} g(x)$ DNE

j. $\lim_{x \to -2^-} g(x) = -2$

c. $\lim_{x \to 9^+} g(x) = -5$

g. $\lim_{x \to 5^+} g(x) = -\infty$

k. $\lim_{x \to -2^+} g(x) = \infty$

d. $\lim_{x \to 9} g(x)$ DNE

h. $\lim_{x \to 3} g(x) = 1$

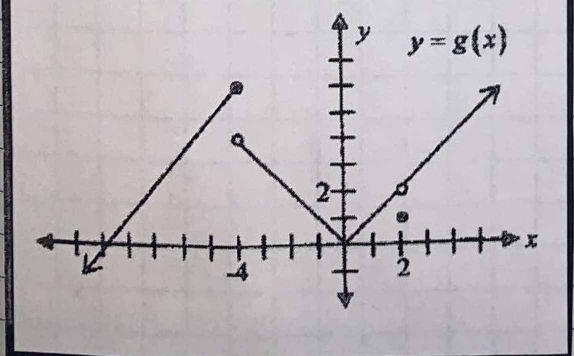
l. $\lim_{x \to -2} g(x)$ DNE

m. $g(3) = -2$ n. $g(-2)$ DNE o. $g(5)$ DNE p. $g(9) = -2$

Consider the function g graphed below. For what values of x_0 does $\lim_{x \to x_0} g(x)$ exist?

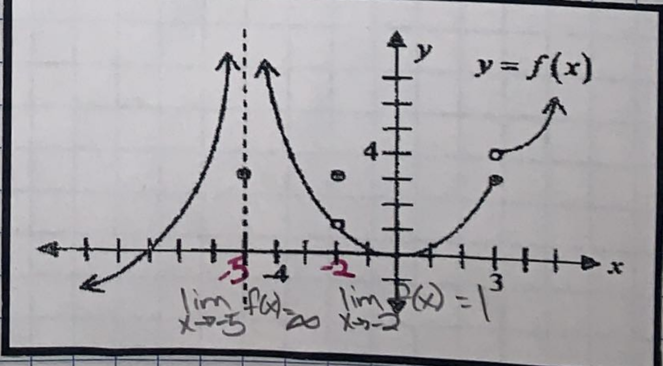
same directions for f(x)

7.



$(-\infty, -4) \cup (-4, \infty)$

8.



$(-\infty, 3) \cup (3, \infty)$

cont. 2.1

Graph from Limits

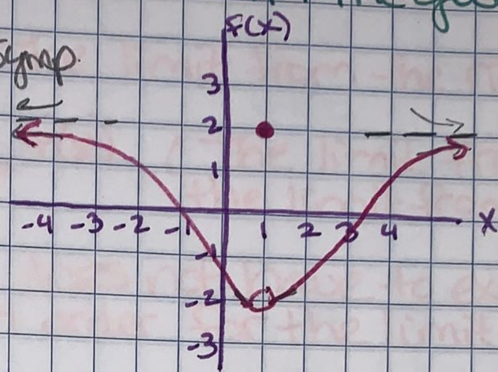
Draw a graph of a function with the given limits:

1. $\lim_{x \to \infty} f(x) = 2$ horiz. asymp.

$\lim_{x \to 1} f(x) = -2$ both sides

$\lim_{x \to -\infty} f(x) = 2$ H.A.

$f(1) = 2$ closed pt @ (1, 2)

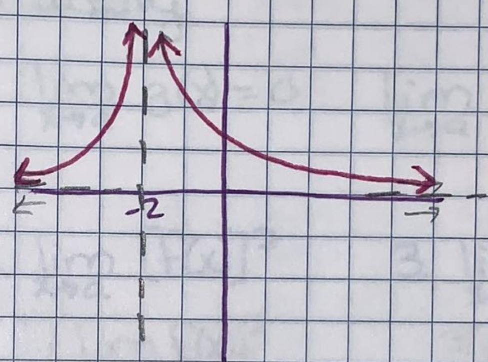


2. $\lim_{x \to \infty} g(x) = 0$ H.A.

$\lim_{x \to -2^+} g(x) = \infty$ VA

$\lim_{x \to -2^-} g(x) = \infty$ VA

$\lim_{x \to -\infty} g(x) = 0$ H.A



2.2 Algebraic Limits

The $\lim_{x\to a} f(x)$ exists IF (and only if)

1. $\lim_{x\to a^-} f(x)$ exists (The limit from the left exists.)
2. $\lim_{x\to a^+} f(x)$ exists (The limit from the right exists.)
3. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ (The limit from the left equals the limit from the right.)

*** The function f(x) does not have to exist at the point (a, f(a)) in order for the limit to exist.

Finding Limits Algebraically

Given: $\lim_{x\to a} f(x) = -3$ $\quad \lim_{x\to a} g(x) = 0$ $\quad \lim_{x\to a} h(x) = 8$

Find:

1. $\lim_{x\to a} [f(x) + h(x)]$

$\lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

$-3 + 8$

5

2. $\lim_{x\to a} [f(x)]^2$

$\left(\lim_{x\to a} f(x)\right)^2$

$(-3)^2$

9

3. $\lim_{x\to a} \sqrt[3]{h(x)}$

$\sqrt[3]{\lim_{x\to a} h(x)}$

$\sqrt[3]{8}$

2

Evaluate the limit:

4. $\lim_{x\to 3} (5x^2 - 6)$

$5(3)^2 - 6 = 39$

5. $\lim_{x\to -1} \dfrac{(x-2)}{x^2+4x-3}$

$\dfrac{-1-2}{(-1)^2+4(-1)-3} = \dfrac{-3}{-6} = \dfrac{1}{2}$

6. $\lim_{x\to 4} \dfrac{x^2-16}{x-4}$

$\dfrac{(4)^2-16}{4-4} = \dfrac{0}{0}$

$\lim_{x\to 4} \dfrac{(x+4)(x-4)}{x-4}$

$\lim_{x\to 4} x+4$

$4+4$

$=8$

7. $\lim_{x\to 0} \dfrac{6x-9}{x^3-12x+3}$

$\dfrac{6(0)-9}{(0)^3-12(0)+3}$

$\dfrac{-9}{3}$

$=-3$

8. $\lim_{x\to 2} \dfrac{x^2-4x+4}{x^2+x-6}$

$\lim_{x\to 2} \dfrac{(x-2)(x-2)}{(x+3)(x-2)}$

$\lim_{x\to 2} \dfrac{x-2}{x+3}$

$\dfrac{2-2}{2+3} = \dfrac{0}{5} = 0$

9. $\lim_{x\to -2} \dfrac{x^3+8}{x+2}$

$\lim_{x\to -2} \dfrac{(x+2)(x^2-2x+4)}{x+2}$

$\lim_{x\to -2} x^2-2x+4$

$(-2)^2-2(-2)+4$

$4+4+4 = 12$

10. $\lim_{x\to -3} \dfrac{x^2-x-12}{x+3}$

$\lim_{x\to -3} \dfrac{(x-4)(x+3)}{x+3}$

$\lim_{x\to -3} x-4$

$-3-4 = -7$

11. $\lim_{x\to 9} \dfrac{9-x}{3-\sqrt{x}}$

$\lim_{x\to 9} \dfrac{(3+\sqrt{x})(3-\sqrt{x})}{(3-\sqrt{x})}$

$\lim_{x\to 9} 3+\sqrt{x}$

$3+\sqrt{9} = 6$

12. $\lim_{x\to 8} \dfrac{x-8}{\sqrt[3]{x}-2}$

$\lim_{x\to 8} \dfrac{(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{\sqrt[3]{x}-2}$

$\lim_{x\to 8} \sqrt[3]{x^2}+2\sqrt[3]{x}+4$

$\sqrt[3]{8^2}+2\sqrt[3]{8}+4$

$4+4+4 = 12$

13. $\lim_{x\to 0} \dfrac{\sqrt{2-x}-\sqrt{2}}{x}$

$\lim_{x\to 0} \dfrac{(\sqrt{2-x}-\sqrt{2})(\sqrt{2-x}+\sqrt{2})}{x(\sqrt{2-x}+\sqrt{2})}$

$\lim_{x\to 0} \dfrac{\cancel{2}-x-\cancel{2}}{x(\sqrt{2-x}+\sqrt{2})}$

$\lim_{x\to 0} \dfrac{-1}{\sqrt{2-x}+\sqrt{2}}$

$\dfrac{-1}{\sqrt{2-0}+\sqrt{2}} = \dfrac{-1}{2\sqrt{2}}$

14. $\lim_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$

$\lim_{x\to 3} \dfrac{\frac{3}{3x}-\frac{x}{3x}}{x-3}$

$\lim_{x\to 3} \dfrac{\frac{-1(x-3)}{3x}}{x-3}$

$\lim_{x\to 3} \dfrac{-1(x-3)}{3x}\cdot\dfrac{1}{x-3}$

$\lim_{x\to 3} \dfrac{-1}{3x} = \dfrac{-1}{3(3)} = \dfrac{-1}{9}$

15. $\lim_{h\to 0} \dfrac{(3+h)^{-1}-3^{-1}}{h}$

$\lim_{h\to 0} \dfrac{\frac{1}{3+h}-\frac{1}{3}}{h}$

$\lim_{h\to 0} \dfrac{\frac{3}{3(3+h)}-\frac{3+h}{3(3+h)}}{h}$

$\lim_{h\to 0} \dfrac{-h}{3(3+h)}\cdot\dfrac{1}{h}$

$\lim_{h\to 0} \dfrac{-1}{3(3+h)}$

$\dfrac{-1}{3(3+0)} = \dfrac{-1}{9}$

16. $\lim_{x\to 2} \frac{(3x-2)^2-(x+2)^2}{x-2}$

$\lim_{x\to 2} \frac{[(3x-2)+(x+2)][(3x-2)-(x+2)]}{x-2}$

$\lim_{x\to 2} \frac{(4x)(2x-4)}{x-2}$

$\lim_{x\to 2} \frac{8x(x-2)}{x-2}$

$\lim_{x\to 2} 8x$

$8(2) = 16$

Definition of Continuity: A function f is continuous at a number a if

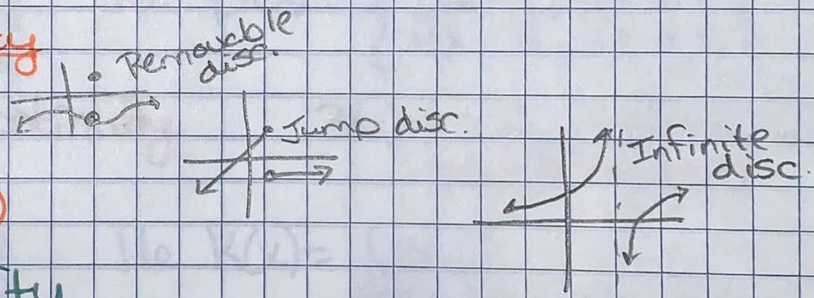
- $f(a)$ is defined
- $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

* Roughly speaking, a function is said to be continuous if it is connected. Can you trace the graph of the function without lifting your finger?

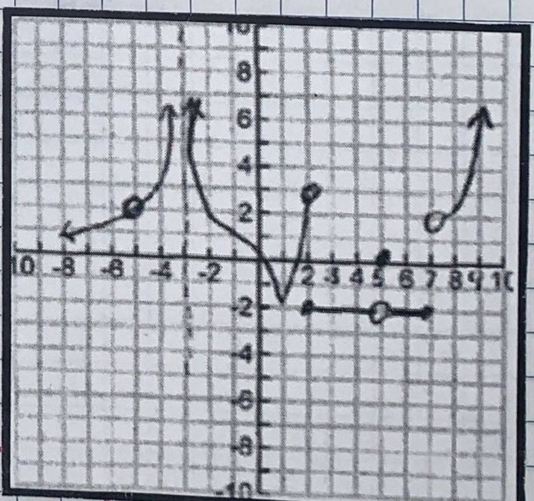
* A function f is right-continuous at $x=a$ if the above is true for limits from the right & left-continuous if the above is true for limits from the left.

Types of Discontinuity

- Removable (hole)
- Jump
- Infinite (asymptote)



Understanding Continuity



1. Does $f(5)$ exist? yes; $f(5)=0$
2. Does $\lim_{x\to 5} f(x)$ exist? yes; $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = -2$
3. Is $f(x)$ continuous at $x=5$? Justify. no; $\lim_{x\to 5} f(x) \neq f(5)$
4. What new value should be assigned to $f(5)$ to remove the discontinuity? $f(5)=-2$
5. Does $f(2)$ exist? yes; $f(2)=-2$
6. Does $\lim_{x\to 2} f(x)$ exist? no; $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$
7. Does $f(-5)$ exist? no; there is a removable discontinuity
8. Does $\lim_{x\to -5} f(x)$ exist? yes; $\lim_{x\to -5^-} f(x) = \lim_{x\to -5^+} f(x) = 2$
9. Is $f(x)$ continuous at $x=-5$? Justify. No; the $\lim_{x\to -5} f(x) \neq f(-5)$
10. What new value should be assigned to $f(-5)$ to make $f(x)$ continuous at $x=-5$? $f(-5)=2$

11. Is $f(x)$ right continuous, left continuous, or neither at $x=2$? How about for $x=7$?

$x=2$ is right continuous because $\lim_{x \to 2^+} f(x) = -2 = f(2)$

$x=7$ is left continuous because $\lim_{x \to 7^-} f(x) = -2 = f(7)$

12. List all places where $f(x)$ is discontinuous + state the type of discontinuity.

$x=-5$ removable discontinuity
$x=-3$ infinite discontinuity
$x=2$ jump discontinuity
$x=5$ removable discontinuity
$x=7$ jump discontinuity

Identify the type of discontinuities in the following:

13. $h(x) = \frac{6}{x-3}$ $\quad x-3 \neq 0$, $x \neq 3$ V.A.

$x=3$ infinite discontinuity

14. $p(x) = \begin{cases} 3x-1, & \text{if } x \geq 1 \\ 4x-2, & \text{if } x < 1 \end{cases}$

$3(1)-1=2 \quad 4(1)-2=2$
continuous

15. $m(x) = \begin{cases} 2x-5, & \text{if } x \geq 2 \\ 3x, & \text{if } x < 2 \end{cases}$

$2(2)-5=-1 \quad 3(2)=6$
jump discontinuity

16. $K(x) = \frac{6x-2}{9x-3}$

$\frac{2\cancel{(3x-1)}}{3\cancel{(3x-1)}}$ hole at $x=\frac{1}{3}$

$x=\frac{1}{3}$ removable disc.

17. $j(x) = \frac{2x-4}{x^2-2x}$

$\frac{2\cancel{(x-2)}}{x\cancel{(x-2)}}$ hole at $x=2$ $\quad x \neq 0$ VA

$x=2$ removable disc.
$x=0$ infinite disc.

Finding Values for Discontinuity

Find a value for a so that $f(x)$ is continuous.

$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ ax+1, & \text{if } x > 2 \end{cases}$

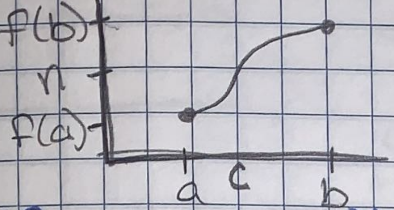
★ must include limit statement!

$2x+3 = ax+1$
$2(2)+3 = a(2)+1$
$7 = 2a+1$
$2a = 6$
$a = 3$

The value that makes this function continuous is $a=3$ because
$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = f(2)$

The Intermediate Value Theorem

If a function f is continuous on $[a,b]$ and n is any number between $f(a)$ and $f(b)$, then there exists at least one c in (a,b) such that $f(c) = n$.



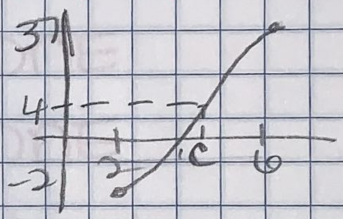
1. Verify the conditions of the IVT + find the guaranteed c value over $[2,6]$ for $f(x) = x^2 + 2x - 11$ when $f(c) = 4$.

Since $f(x)$ is continuous from $(2,6)$,

$f(2) = 2^2 + 2(2) - 11 = -3$

$f(6) = 6^2 + 2(6) - 11 = 37$

and $-3 < 4 < 37$



$\therefore \exists c \in (2,6)$ s.t. $f(c) = 4$

$f(c) = c^2 + 2c - 11 = 4$

$c^2 + 2c - 15 = 0$

$(c+5)(c-3) = 0$

$\cancel{c = -5}$ $\quad c = 3$

not b/n $(2,6)$ $\quad c = 3$

Key for Symbols

- $\therefore$ "therefore"
- $\exists$ "there exists"
- $\in$ "an element of (in)"
- s.t. "such that"

2. Use the IVT to show that $f(x) = x^3 + 5$ has a root in the interval $(-2,-1)$

Since $f(x)$ is continuous from $(-2,-1)$

$f(-2) = (-2)^3 + 5 = -3$

$f(-1) = (-1)^3 + 5 = 4$

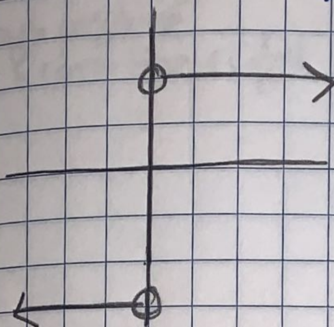
and $-3 < 0 < 4$

$\therefore \exists c \in (-2,-1)$ s.t. $f(c) = 0$

2.4 One-Sided Limits

Find the limit graphically

1. $G(x)=\dfrac{|3x|}{x}$ $\qquad G(x)=\begin{cases}\dfrac{3x}{x}, & x>1\\ \dfrac{-3x}{x}, & x<1\end{cases}$



x	y
2	3
1	3
0	DNE
-1	-3
-2	-3

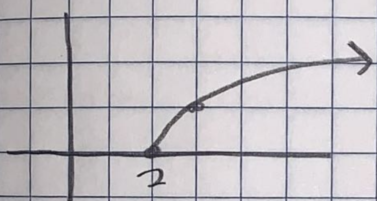
a. $\lim_{x\to 0^+} G(x) = 3$

b. $\lim_{x\to 0^-} G(x) = -3$

c. $\lim_{x\to 0} G(x)$ DNE

d. $G(0)$ DNE

2. $H(x)=\sqrt{x-2}$



a. $\lim_{x\to 2^+} H(x) = 0$

b. $\lim_{x\to 2^-} H(x)$ DNE

c. $\lim_{x\to 2} H(x)$ DNE

d. $H(2) = 0$

Find the limit algebraically.

$$F(x)=\begin{cases}x^2-16, & \text{if } x<3\\ 5, & \text{if } x=3\\ 14-x^2, & \text{if } x>3\end{cases}$$

a. $\lim_{x\to 3^+} f(x) = 14-(3)^2 = 5$

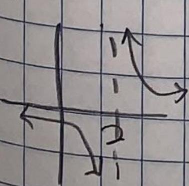
b. $\lim_{x\to 3^-} f(x) = (3)^2-16 = -7$

c. $\lim_{x\to 3} f(x)$ DNE

d. $f(3) = 5$

Find the limits algebraically:

a. $\lim_{x\to 2^-} \dfrac{3}{x-2} = -\infty$



b. $\lim_{x\to 5^-} \dfrac{15-3x}{|4x-20|}$

$\lim_{x\to 5^-} \dfrac{-3(x-5)}{4|x-5|}$

$\lim_{x\to 5^-} \dfrac{-3\cancel{(x-5)}}{-4\cancel{(x-5)}} = \dfrac{3}{4}$

bc it's from the left

cont. 2.4

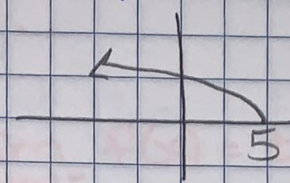
c. $\lim_{x \to 2^+} \frac{|x-2|}{x-2}$

$\lim_{x \to 2^+} \frac{x-2}{x-2} = 1$

stays positive bc from the right

d. $\lim_{x \to 5^+} \sqrt{5-x}$

reflects over y-axis + shifts right 5



DNE

2.5 Limits Involving Infinity

Vertical and Horizontal Asymptotes

Formal definitions of:

a. Vertical Asymptotes:

$\lim_{x \to a^+} f(x) = \pm\infty \qquad \lim_{x \to a^-} f(x) = \pm\infty$

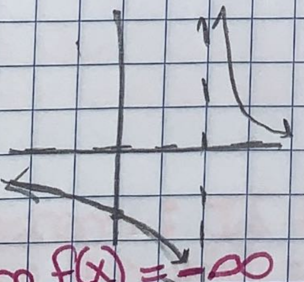
b. Horizontal Asymptote: $\lim_{x \to \pm\infty} f(x) = b$

How to find asymptotes:

- Vertical Asymptotes
  1. Factor numerator + denominator to remove any removable discontinuities
  2. Set the denominator equal to zero + solve.
- Horizontal Asymptotes
  - Case 1: Degree of Numerator = Degree of Denominator then $y = \frac{\text{lead coeff. numerator}}{\text{lead coeff. denominator}}$
  - Case 2: Degree of Numerator < Degree of Denom. then $y = 0$
  - Case 3: Degree of Numerator > Degree of Denominator then no H.A. (goes off to $\pm\infty$)
    - ★ It has a slant asymptote if degree in numerator is 1 higher than deg. of denom.

Asymptotes

1. $y = \frac{3}{x-2}$



V.A: $x = 2$

Justification:

$\lim_{x \to 2^+} f(x) = \infty \qquad \lim_{x \to 2^-} f(x) = -\infty$

H.A.: $y = 0$

Justification: $\lim_{x \to \pm\infty} f(x) = 0$

2. $f(x) = \frac{x+3}{x^2+x-6} = \frac{\cancel{x+3}}{(\cancel{x+3})(x-2)}$ removable disc. @ $x = -3$

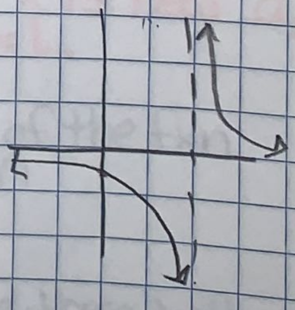
$f(x) = \frac{1}{x-2}$

V.A.: $x = 2$

Justification:

$\lim_{x \to 2^+} f(x) = \infty$

$\lim_{x \to 2^-} f(x) = -\infty$



H.A.: $y = 0$

Justification:

$\lim_{x \to \pm\infty} f(x) = 0$

cont. 2.5

1. $\lim_{x\to\infty} \dfrac{4x^2+5x}{6-3x^1} = \infty$

compare degrees

2. $\lim_{x\to\infty} \dfrac{x^1-4}{x^2-2x+5} = 0$

3. $\lim_{x\to-\infty} \dfrac{6x^2+5x}{(1-x)(2x-3)}$

behaves like

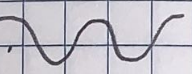
$\lim_{x\to-\infty} \dfrac{6x^2}{-2x^2} = -3$

4. $\lim_{x\to\infty} \dfrac{\sqrt{1+4x^2}}{4+x}$ $\qquad \dfrac{\sqrt{4x^2}}{x}$

behaves like

$\lim_{x\to\infty} \dfrac{2x}{x} = 2$

5. $\lim_{x\to\infty} \cos(x)$ DNE



oscillates at an even height

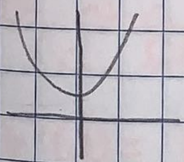
6. $\lim_{x\to-\infty} (3-x) = \infty$

7. $\lim_{x\to\infty} (1+2x-3x^2) = -\infty$

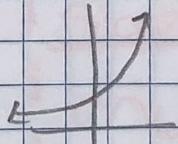
Think about end behavior

8. $\lim_{x\to-\infty} (3-x^2) = -\infty$

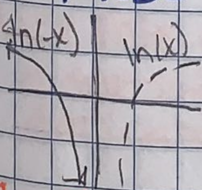
9. $\lim_{x\to\infty} e^{|x|} = \infty$



10. $\lim_{x\to\infty} e^x = \infty$



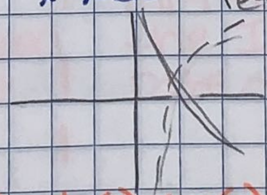
11. $\lim_{x\to-\infty} \ln(-x) = \infty$



reflect over y-axis

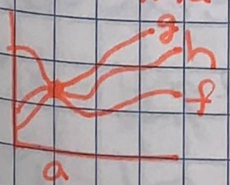
12. $\lim_{x\to-\infty} (-\ln(x))$ DNE

reflect over x-axis



V.A. at $x=0$

The Squeeze Theorem: If $f(x) < h(x) < g(x)$ when x is near a and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, then $\lim_{x\to a} h(x) = L$



use when you don't know the graph of the fxn but you know 2 graphs it's in between

$\lim_{x\to\infty} \left(\dfrac{\sin x}{x}\right) = 0$

$-1 \le \sin x \le 1$

$\dfrac{-1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$

$\lim_{x\to\infty} -\dfrac{1}{x} = 0$

$\lim_{x\to\infty} \dfrac{1}{x} = 0$

Since these both equal 0 & $\frac{\sin x}{x}$ is between them, then it has the same limit