Keeper 1.4 - Composition, Exponentials and Logs

The Algebra of Functions

$(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$

Composition of Functions

Given two functions f and g where x is in the domain of g and $g(x)$ is in the domain of f then,

$(f \circ g)(x) = f(g(x))$

and

$(g \circ f)(x) = g(f(x))$

Examples: Evaluate the Functions

1.

x	-2	-1	0	1	2	3
$f(x)$	-3	-2	1	4	-1	0
$g(x)$	-2	0	1	3	-1	2

a. $(f+g)(-1)$

$= f(-1) + g(-1)$

$= -2 + 0$

$= -2$

b. $(f \circ g)(3)$

$= f(g(3))$

$= f(2)$

$= -1$

c. $(g \circ g)(1)$

$= g(g(-1))$

$= g(0)$

$= 1$

d. $g^{-1}(0)$ ← y value

* What x value gives a y value of 0 for g

$g^{-1}(0) = -1$

e. $f^{-1}(-2)$

↖ y value

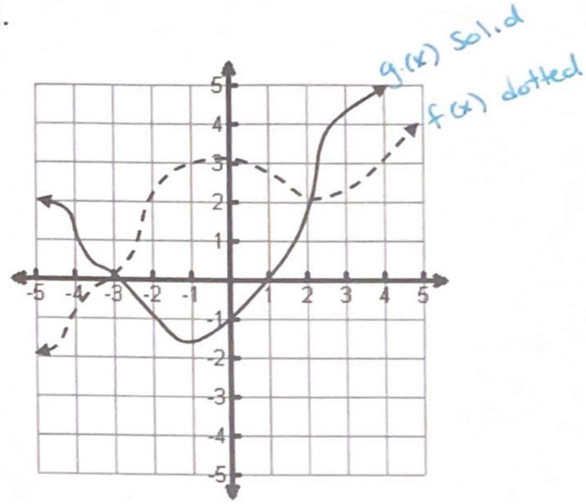
$f^{-1}(-2) = -1$

f. $f(g(-1))$

$= f(0)$

$= 1$

2.



a. $(f-g)(4) = f(4) - g(4) = 3 - 5 = -2$

b. $(g-f)(-2) = g(-2) - f(-2) = -1 - 2 = -3$

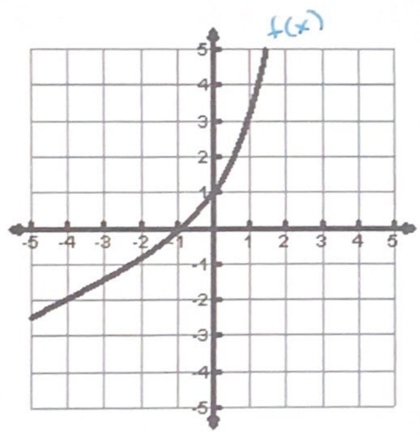
c. $(f \circ g)(1) = f(g(1)) = f(0) = 3$

d. $(g \circ f)(-2) = g(f(-2)) = g(2) = 2$

e. $(f \circ f)(-3) = f(f(-3)) = f(0) = 3$

f. $(g \circ g)(-4) = g(g(-4)) = g(1) = 0$

3.



Find:

$f(0) = 0$ $\quad f^{-1}(3) = 1$

↓
y value

$f(1) = 3$ $\quad f^{-1}(1) = 0$

↓
y value

$f(-2) = -1$ $\quad f^{-1}(-2) = -4$

↓
y value

Logarithmic Properties	
Product Rule	$\log_a(xy) = \log_a x + \log_a y$
Quotient Rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power Rule	$\log_a x^p = p\log_a x$
Change of Base Rule	$\log_a x = \frac{\log_b x}{\log_b a}$
Equality Rule	If $\log_a x = \log_a y$ then $x = y$

Examples.

4. Find the value of the following without using a calculator.

   a. $3\ln e + \ln\left(\frac{1}{e}\right)$

   $= 3(1) + \ln e^{-1}$

   $= 3 + (-1)$

   $= 2$

   b. $\ln e^2 + e^{-\ln e}$

   $= 2\ln e + e^{\ln e^{-1}}$

   $= 2 + e^{-1}$

   $= 2 + \frac{1}{e}$

   * Do Not Leave Negative exponents!

   c. $e^{-\ln\sqrt{e}}$

   $= e^{\ln e^{-1/2}}$

   $= e^{-1/2}$

   $= \frac{1}{\sqrt{e}}$

   * No Need to Rationalize

5. Simplify to a single ln or e expression, or to a single number.

   a. $2\ln a - 3\ln b + \ln(ab)$

   $= \ln a^2 - \ln b^3 + \ln(ab)$

   $= \ln \frac{a^2}{b^3}(ab)$

   $= \ln\left(\frac{a^3}{b^2}\right)$

   b. $\ln(xe^{-\ln x})$

   $= \ln(xe^{\ln x^{-1}})$

   $= \ln(x \cdot x^{-1})$

   $= \ln(1)$

   $= 0$

   * Remember $\ln e = 1$

   c. $\ln(e^2\ln(e\ln e))$

   $= \ln(e^2\ln(e))$

   $= \ln(e^2)$

   $= 2\ln e$

   $= 2$

6. Express as a logarithm of a single expression.

   a. $2\ln x + 4\ln y - \ln 13$

   $= \ln x^2 + \ln y^4 - \ln 13$

   $= \ln\left(\frac{x^2y^4}{13}\right)$

   b. $\ln 7 + 5\ln y - \frac{\ln x}{2}$

   $\ln 7 + \ln y^5 - \ln x^{1/2}$

   $\ln\left(\frac{7y^5}{\sqrt{x}}\right)$

   c. $\log(x^2 - 16) - 3\log(x + 4) + 2\log x$

   $\log(x^2-16) - \log(x+4)^3 + \log x^2$

   $\log\left(\frac{x^2(x^2-16)}{(x+4)^3}\right)$

   * Simplify!

   $\log\left(\frac{x^2(x-4)(x+4)}{(x+4)^3}\right)$

   $\log\left(\frac{x^2(x-4)}{(x+4)^2}\right)$

7. Expand the following logarithms.

   a. $\log\frac{x^2y^3}{z}$

   $\log x^2 + \log y^3 - \log z$

   $2\log x + 3\log y - \log z$

   b. $\ln\frac{x\sqrt[3]{4x+1}}{\sqrt{2x-1}}$

   $\ln x + \ln\sqrt[3]{4x+1} - \ln\sqrt{2x-1}$

   $\ln x + \ln(4x+1)^{1/3} - \ln(2x-1)^{1/2}$

   $\ln x + \frac{1}{3}\ln(4x+1) - \frac{1}{2}\ln(2x-1)$

   c. $\log\frac{z}{\sqrt{xy}}$

   $\log z - \log\sqrt{xy}$

   $\log z - \frac{1}{2}\log(xy)$

   $\log z - \frac{1}{2}\log x - \frac{1}{2}\log y$

   or

   $\log z - \frac{1}{2}(\log x + \log y)$

Solve the following equations

8. $2^x + 5 = 17$

$2^x = 12$

$x = \log_2 12$

9. $7.01 = 3^{2x} - 2.4$

$3^{2x} = 9.41$

$2x = \log_3 9.41$

$x = \dfrac{\log_3 9.41}{2}$

10. $5e^{x+1} = 27$

$e^{x+1} = \frac{27}{5}$

$x+1 = \ln\left(\frac{27}{5}\right)$

$x = \ln\left(\frac{27}{5}\right) - 1$

11. $3 \cdot 2^{2x+1} = 24$

$2^{2x+1} = 8$

$2^{2x+1} = 2^3$

$2x+1 = 3$

$2x = 2$

$x = 1$

12. $e^{x^2+2x} = 2$

$x^2+2x = \ln 2$

$x^2+2x-\ln 2 = 0$

$x = \dfrac{-2 \pm \sqrt{4+4\ln 2}}{2}$

$x = -1 \pm \sqrt{1+\ln 2}$

13. $4e^{2x-3} - 5 = e$

$4e^{2x-3} = e+5$

$e^{2x-3} = \frac{e+5}{4}$

$2x-3 = \ln\left(\frac{e+5}{4}\right)$

$2x = \ln\left(\frac{e+5}{4}\right)+3$

$x = \dfrac{\ln\left(\frac{e+5}{4}\right)+3}{2}$

14. $2P = Pe^{kx}$

$2 = e^{kx}$

$\ln 2 = kx$

$x = \dfrac{\ln 2}{k}$

15. $ae^{kx} = eb^{kx}$

$\dfrac{e^{kx}}{b^{kx}} = \dfrac{e}{a}$

$\left(\dfrac{e}{b}\right)^{kx} = \dfrac{e}{a}$

$kx \ln\left(\frac{e}{b}\right) = \ln\left(\frac{e}{a}\right)$

$kx(1-\ln b) = 1-\ln a$

$x = \dfrac{1-\ln a}{k(1-\ln a)}$

16. $3(3^x) - 5x(3^x) + 2x^2(3^x) = 0$

$3^x(2x^2-5x+3) = 0$

$3^x(x-1)(2x-3) = 0$

$3^x = 0$ DNE $\quad x-1=0$, $x=1$ $\quad 2x-3=0$, $x=3/2$

$x = 1, 3/2$

17. $\ln(x-2) = 2\ln(x)$

$\ln(x-2) = \ln x^2$

$x-2 = x^2$

$x^2-x+2 = 0$

$x = \dfrac{1 \pm \sqrt{1-4(1)(2)}}{2}$

$x = \dfrac{1 \pm \sqrt{7}i}{2}$

Imaginary!

18. $\ln(2x^2-4) = 5$

$2x^2-4 = e^5$

$2x^2 = e^5+4$

$x^2 = \dfrac{e^5+4}{2}$

$x = \pm\sqrt{\dfrac{e^5+4}{2}}$

19. $\log_3(\log_3(2x)) = 1$

$\log_3(2x) = 3$

$2x = 3^3$

$2x = 27$

$x = \dfrac{27}{2}$

Evaluate the expressions with no calculator.

20. $36^{\frac{1}{2}}$

$= \sqrt{36}$

$= 6$

21. $8^{\frac{2}{3}}$

$= (\sqrt[3]{8})^2$

$= 2^2$

$= 4$

22. $\left(\frac{4}{9}\right)^{-\frac{1}{2}}$

$= \left(\frac{9}{4}\right)^{1/2}$

$= \sqrt{9/4}$

$= \frac{\sqrt{9}}{\sqrt{4}}$

$= \frac{3}{2}$

23. $\left(\frac{64x^{27}}{125y^{-8}}\right)^{-\frac{1}{3}}$

$= \left(\frac{125y^{-8}}{64x^{27}}\right)^{1/3}$

$= \left(\frac{125}{64x^{27}y^{8}}\right)^{1/3}$

$= \frac{125^{1/3}}{64^{1/3}x^{9}y^{8/3}}$

$= \frac{5}{4x^9y^{8/3}}$

24. $\left(\frac{1000}{27}\right)^{-\frac{2}{3}}$

$= \left(\frac{27}{1000}\right)^{2/3}$

$= \sqrt[3]{\frac{27}{1000}}^{\,2}$

$= \left(\frac{3}{10}\right)^2$

$= \frac{9}{100}$

25. $(64)^{-\frac{3}{2}}$

$\left(\frac{1}{64}\right)^{3/2}$

$\left(\sqrt{1/64}\right)^3$

$\left(\frac{1}{8}\right)^3$

$\frac{1}{512}$