Quick Indefinite Integrals (6.1)

Power Rule Shortcut "Reverse Chain Rule"

- Keep inside
- Add 1 to the exponent + divide by that exponent
- Divide by the derivative of the inside

1. $\int (5x+3)^4\,dx = \frac{(5x+3)^{4+1}}{5\cdot 5} = \frac{(5x+3)^5}{25} + C$
2. $\int (1-8x)^{99}\,dx = \frac{(1-8x)^{100}}{100\cdot -8} = \frac{(1-8x)^{100}}{-800} + C$
3. $\int (2x-7)^{3/4}\,dx = \frac{\cancel{4}(2x-7)^{7/4}}{7\cdot \cancel{2}} = \frac{2(2x-7)}{7} + C$
4. $\int \sqrt{3x+1}\,dx = \frac{\frac{2}{3}(3x+1)^{1/2+1}}{3} + C = \frac{2(3x+1)^{3/2}}{9} + C$

Exponential Rule Shortcut (base e)

- Copy the problem
- Divide by ln base (which $\ln e = 1$)
- Divide by the derivative of the exponent

1. $\int e^{-x}\,dx = \frac{e^{-x}}{-1} + c = -e^{-x} + c = \frac{-1}{e^x} + C$
2. $\int -e^{5x+12}\,dx = \frac{-e^{5x+12}}{5} + C$
3. $\int 3e^{5x}\,dx = \frac{3e^{5x}}{5} + C$
4. $\int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}\frac{e^{2x}}{2} + c$

   $\frac{e^{2x}}{4} + C$

Exponential Shortcut (not base e)

- Copy the problem
- Divide by ln of the base (don't ignore it)
- Divide by the derivative of the exponent

1. $\int -2^x\,dx = \frac{-2^x}{\ln 2} + C$
2. $\int 2^{5x+12}\,dx = \frac{2^{5x+12}}{5\ln 2} + C$
3. $\int 7^{6-5x} = \frac{7^{6-5x}}{-5\ln 7} + C$
4. $\int 3^{2x}\,dx = \frac{3^{2x}}{2\ln 3} + C$

Trig Shortcut "Reverse Chain Rule"

- Integrate the 'outside' (trig integrals)
- Divide by the derivative of the 'inside'!

1. $\int \cos \frac{x}{2}\,dx = \frac{\sin \frac{x}{2}}{1/2} + C = 2\sin\frac{x}{2} + C$

2. $\int \sec \pi x \tan \pi x\,dx = \frac{\sec \pi x}{\pi} + C$

3. $\int \csc 3x \cot 3x\,dx = -\frac{\csc 3x}{3} + C$

4. $\int \sec^2(2x-3)\,dx = \frac{\tan(2x-3)}{2} + C$

5. $\int \frac{\sin 15x}{3} = -\frac{\cos 15x}{3 \cdot 15} + C = -\frac{\cos 15x}{45} + C$

$\frac{1}{x}$ or x^{-1} Shortcut

$\int \frac{1}{x}$ or $\int x^{-1}\,dx = \ln|x|$

- ln|function|
- divided by the derivative of the function

1. $\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln|x| + C$

2. $\int 8x^{-1}\,dx = 8\int x^{-1}\,dx = 8\ln|x| + C$

3. $\int \frac{1}{4x}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx = \frac{1}{4}\ln|x| + C$

4. $\int -\frac{1}{2}x^{-1}\,dx = -\frac{1}{2}\int x^{-1}\,dx = -\frac{1}{2}\ln|x| + C$

5. $\int \frac{1}{x+5}\,dx = \ln|x+5| + C$

6. $\int \frac{4}{2x-1}\,dx = 4\int \frac{1}{2x-1}\,dx = \frac{4\ln|2x-1|}{2} = 2\ln|2x-1| + C$

7.

U-Substitution - Indefinite Integrals (6.2)

This method undoes a chain rule derivative!

Steps for integrating with u-substitution:

1. Define u (usually a piece of the function INSIDE the function or in the Denominator or the Exponent)
2. Find derivative of u with respect to x ($\frac{du}{dx}$)
3. Solve for dx
4. Substitute u into the problem + substitute in for dx.
5. Simplify + bring out any constants to the front.
6. Integrate
7. Substitute back in for u.

We learned how to do this one as a quick integral but we are going to see how to integrate it with u-sub:

$$\int (5x+3)^4\,dx$$

$$\int u^4\,dx \qquad u = 5x+3$$

$$\int u^4\,\frac{du}{5} \qquad du = 5\,dx$$

$$dx = \frac{du}{5}$$

$$\frac{1}{5}\int u^4\,du = \frac{1}{5}\cdot\frac{u^5}{5} + C = \frac{u^5}{25} + C = \frac{(5x+3)^5}{25} + C$$

1. $\int (3x-4)e^{\frac{3}{2}x^2-4x}\,dx$

$$\int (3x-4)e^u\,dx \qquad u = \tfrac{3}{2}x^2 - 4x$$

$$\int \cancel{(3x-4)}\,e^u\,\frac{du}{\cancel{(3x-4)}} \qquad du = 3x-4\,dx$$

$$dx = \frac{du}{3x-4}$$

$$\int e^u\,du$$

$$e^u + C$$

$$e^{\frac{3}{2}x^2-4x} + C$$

2. $\int x\cos(x^2)\,dx$

$$\int x\cos u\,dx \qquad u = x^2$$

$$\frac{1}{2}\int \cancel{x}\cos u\,\frac{du}{\cancel{2x}} \qquad du = 2x\,dx$$

$$dx = \frac{du}{2x}$$

$$\frac{1}{2}\int \cos u\,du$$

$$\frac{1}{2}\sin u + C$$

$$\frac{1}{2}\sin x^2 + C$$

or $\frac{\sin x^2}{2} + C$

3. $\int \frac{8x}{e^{x^2}}\,dx$

$\int \frac{8x}{e^u}\,dx$ $\quad u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$\int \frac{\overset{4}{\cancel{8x}}}{e^u}\cdot\frac{du}{\cancel{2x}}$

$4\int e^{-u}\,du$

$4\frac{e^{-u}}{-1} + C = \frac{-4}{e^u} + C$

$\frac{-4}{e^{x^2}} + C$

4. $\int \frac{3x}{5-4x^2}\,dx$

$\int \frac{3x}{u}\,dx$ $\quad u = 5-4x^2$

$du = -8x\,dx$

$dx = \frac{du}{-8x}$

$\int \frac{3\cancel{x}}{u}\,\frac{du}{-8\cancel{x}}$

$\frac{-3}{8}\int \frac{1}{u}\,du$

$\frac{-3}{8}\ln|u| + C$

$\frac{-3\ln|5-4x^2|}{8} + C$

5. $\int x\sqrt[3]{x^2+5}\,dx$

$\int x\sqrt[3]{u}\,dx$ $\quad u = x^2+5$

$du = 2x$

$dx = \frac{du}{2x}$

$\int \cancel{x}\,u^{1/3}\,\frac{du}{2\cancel{x}}$

$\frac{1}{2}\int u^{1/3}\,du$

$\frac{1}{2}\,\frac{u^{4/3}\cdot 3}{4} + C = \frac{3u^{4/3}}{8} + C$

$\frac{3(x^2+5)^{4/3}}{8} + C$

6. $\int 7x^2\cos(4x^3)\,dx$

$\int 7x^2\cos u\,dx$ $\quad u = 4x^3$

$du = 12x^2\,dx$

$dx = \frac{du}{12x^2}$

$\int 7\cancel{x^2}\cos u\,\frac{du}{12\cancel{x^2}}$

$\frac{7}{12}\int \cos u\,du$

$\frac{7}{12}\sin u + C$

$\frac{7}{12}\sin(4x^3) + C$

U-Substitution: Definite Integrals (6.3)

Change of Variables for Definite Integrals –
If the function $y=g(x)$ has a continuous derivative on the closed interval $[a,b]$ + f is continuous on the range g, then $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

1. $\int_0^1 \sqrt{t^5+3t}\,(5t^4+3)\,dt$ $\boxed{\text{2 Ways!}}$

$\int_0^1 \sqrt{u}\,(5t^4+3)\,dt$

$\int_0^1 u^{1/2}\cancel{(5t^4+3)}\dfrac{du}{\cancel{(5t^4+3)}}$

$u=t^5+3t$
$du=5t^4+3\,dt$
$dt=\dfrac{du}{5t^4+3}$

$u=t^5+3t$
$u_1=4$
$u_0=0$
bounds for u

$\dfrac{2u^{3/2}}{3}+C$

you changed variables so you need to change bounds

$\dfrac{2u^{3/2}}{3}\Big|_0^4 = \left(\dfrac{2\sqrt{4}^3}{3} - \dfrac{2\sqrt{0}^3}{3}\right) = \dfrac{16}{3}$

or you can plug t^5+3t back in for u + use the original upper + lower bounds.

$\dfrac{2(t^5+3t)^{3/2}}{3}\Big|_0^1 = \dfrac{2(1^5+3(1))^{3/2}}{3} - \dfrac{2(0^5+3(0))^{3/2}}{3} = \dfrac{16}{3}$

2. $\int_0^{\pi/3} \tan x \sec^2 x\,dx$

$\int_0^{\pi/3} u\sec^2 x\,dx$

$\int_0^{\pi/3} u\,\cancel{\sec^2 x}\,\dfrac{du}{\cancel{\sec^2 x}}$

$u=\tan x$
$du=\sec^2 x\,dx$
$dx=\dfrac{du}{\sec^2 x}$

$u=\tan x$
$u_{\pi/3}=\tan\frac{\pi}{3}=\sqrt{3}$
$u_0=\tan 0=0$

$\dfrac{u^2}{2}\Big|_0^{\sqrt{3}} = \dfrac{(\sqrt{3})^2}{2} - \dfrac{(0)^2}{2} = \dfrac{3}{2}$

3. $\int_0^1 x^2(x^3+1)^4\,dx$

$\int_0^1 x^2 u^4\,dx$ $u = x^3+1$ $u_1 = 2$

$\int_0^1 x^2 u^4 \frac{du}{3x^2}$ $du = 3x^2\,dx$ $u_0 = 1$

$dx = \frac{du}{3x^2}$

$\frac{1}{3}\int u^4\,du$

$\frac{1}{3}\cdot\frac{u^5}{5}\Big|_1^2 = \frac{(2)^5}{15} - \frac{(1)^5}{15} = \frac{31}{15}$

4. $\int_\pi^{\pi/2} \cos^3 x \sin x\,dx$

$\int_\pi^{\pi/2} u^3 \sin x\,dx$ $u = \cos x$ $u = \cos x$

$du = -\sin x\,dx$ $u_{\pi/2} = 0$

$\int_\pi^{\pi/2} u^3 \sin x \frac{du}{-\sin x}$ $dx = \frac{du}{-\sin x}$ $u_\pi = -1$

$-\frac{u^4}{4}\Big|_{-1}^0 = -\frac{0^4}{4} - -\frac{(-1)^4}{4} = \frac{1}{4}$

5. $\int_0^1 x^3 e^{x^4}\,dx$

$\int_0^1 x^3 e^u\,dx$ $u = x^4$ $u = x^4$

$du = 4x^3\,dx$ $u_1 = 1$

$\int_0 x^3 e^u \frac{du}{4x^3}$ $dx = \frac{du}{4x^3}$ $u_0 = 0$

$\frac{1}{4}\int_0^1 e^u\,du$

$\frac{e^u}{4}\Big|_0^1 = \frac{e^1}{4} - \frac{e^0}{4} = \frac{e}{4} - \frac{1}{4}$ or $\frac{e-1}{4}$

Integration By Parts - No Table (6.4)

Undoes the Product Rule : $\int uv' = uv - \int v\,du$

How to choose $u \rightarrow$ L. I. A. T. E.

① Logs (ln) ② Inverse Trig ③ Algebraic ④ Trig ⑤ Exponents

1. $\int \overset{A}{x}\,\overset{E}{e^x}\,dx$ $\quad u = x \quad dv = e^x$

$xe^x - \int e^x \cdot 1dx$ $\quad du = 1dx \quad v = e^x$

$xe^x - e^x + C$

2. $\int \overset{A}{x^6} \cdot \overset{L}{\ln} x\,dx$ $\quad u = \ln x \quad dv = x^6$

$\ln x \cdot \frac{x^7}{7} - \int \frac{x^7}{7} \cdot \frac{1}{x}dx$ $\quad du = \frac{1}{x}dx \quad v = \frac{x^7}{7}$

$\frac{x^7 \ln x}{7} - \frac{1}{7}\int x^6 dx = \frac{x^7 \ln x}{7} - \frac{1}{7}\frac{x^7}{7} + C = \frac{x^7 \ln x}{7} - \frac{x^7}{49} + C$

3. $\int \overset{A}{x^2} \cdot \overset{T}{\sin} 4x\,dx$ $\quad u = x^2 \quad dv = \sin 4x$

$-\frac{x^2\cos 4x}{4} - \int \frac{-\cos 4x}{\cancel{4}_{2}} \cdot \cancel{2}x\,dx$ $\quad du = 2x\,dx \quad v = \frac{-\cos 4x}{4}$

$-\frac{x^2\cos 4x}{4} + \frac{1}{2}\int \overset{T}{\cos} 4x \cdot \overset{A}{x}\,dx$ Integrate by parts again

$-\frac{x^2\cos 4x}{4} + \frac{1}{2}\left(x\frac{\sin 4x}{4} - \int \frac{\sin 4x}{4} \cdot 1dx\right)$ $\quad u = x \quad dv = \cos 4x$; $\quad du = 1dx \quad v = \frac{\sin 4x}{4}$

$-\frac{x^2\cos 4x}{4} + \frac{x\sin 4x}{8} - \frac{1}{8}\int \sin 4x\,dx$

$-\frac{x^2\cos 4x}{4} + \frac{x\sin 4x}{8} + \frac{\cos 4x}{32} + C$

4. $\int \frac{\ln x}{x^2}dx$ Rewrite as a product 1st!

$\int \overset{L}{\ln x} \cdot \overset{A}{x^{-2}}\,dx$ $\quad u = \ln x \quad dv = x^{-2}$

$\ln x \cdot \frac{-1}{x} - \int \frac{-1}{x} \cdot \frac{1}{x}dx$ $\quad du = \frac{1}{x}dx \quad v = \frac{x^{-1}}{-1} = \frac{-1}{x}$

$-\frac{\ln x}{x} + \int x^{-2}dx$

$-\frac{\ln x}{x} - \frac{1}{x} + C$

Integration By Parts – With Table (6.5)

*This method can **only** be used when u is algebraic

Steps:
1. Set up a table with a 'u' column & a 'dv' column
2. Find the derivative of u until you get to 0
3. Find the antiderivative of dv until you get next to 0
4. Label each row with alternating signs (+, −, +, − etc
5. Multiply diagonally from left to right going down

A T

1. $\int x^2 \sin(4x)\,dx$ (same as #3 yesterday)

	u	dv
+	x^2	$\sin(4x)$
−	$2x$	$-\frac{\cos(4x)}{4}$
+	2	$-\frac{\sin(4x)}{16}$
−	0	$\frac{\cos(4x)}{64}$

$$\frac{-x^2\cos(4x)}{4} + \frac{\sin(4x)}{8} + \frac{\cos(4x)}{32} + C$$

E A

2. $\int e^x x^4\,dx$

	u	
+	x^4	e^x
	$-4x^3$	e^x
	$+12x^2$	e^x
	$-24x$	e^x
	$+24$	e^x
	-0	e^x

$$x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$$

A T

3. $\int 5x^3\cos(2x)\,dx$

	u	dv
+	$5x^3$	$\cos(2x)$
−	$15x^2$	$\frac{\sin(2x)}{2}$
+	$30x$	$-\frac{\cos(2x)}{4}$
−	30	$-\frac{\sin(2x)}{8}$
+	0	$\frac{\cos(2x)}{16}$

$$\frac{5x^3\sin 2x}{2} + \frac{15x^2\cos 2x}{4} - \frac{15\sin 2x}{4} - \frac{15\cos 2x}{8} + C$$

A T

4. $\int (x^2+2x+1)\sin x\,dx$

	u	dv
+	x^2+2x+1	$\sin x$
−	$(2x+2)$	$-\cos x$
+	2	$-\sin x$
−	0	$\cos x$

$$-(x^2+2x+1)\cos x + (2x+2)\sin x + 2\cos x + C$$

Differential Equations (6.6)

an equation containing derivatives

Separation of variables method for solving:

- Rewrite the equations so that each variable occurs on only one side of the equation.
- Integrate both sides of the equation separately.
- Solve for y.

Solve the differential equations (Find the general solution.)

1. $\frac{dy}{dx} = 4x - 5$

$\int dy = \int 4x - 5\, dx$

$y = \frac{4x^2}{2} - 5x + C$

$y = 2x^2 - 5x + C$

2. $\frac{dy}{dx} = \cos x + 5$

$\int dy = \int \cos x + 5\, dx$

$y = \sin x + 5x + C$

3. $\frac{dy}{dx} = \frac{2x}{y}$

$\int y\, dy = \int 2x\, dx$

$\frac{y^2}{2} = \frac{2x^2}{2} + C$

$y^2 = 2x^2 + C$

$y = \pm\sqrt{2x^2 + C}$

4. $\frac{dy}{dx} = \sqrt{x}\, y$

$\int \frac{1}{y}\, dy = \int \sqrt{x}\, dx$

$\ln|y| = \frac{2x^{3/2}}{3} + C$

$\log_e|y|$

$y = e^{2/3x^{3/2} + C}$ Prop. of Exp.

Just a constant

$y = e^C \cdot e^{2/3x^{3/2}}$

$y = C \cdot e^{2/3x^{3/2}}$

Find the particular solution

5. $\frac{dy}{dx} = x^2 - 2x - 4$; $y = -6$ when $x = 3$

$\int dy = \int x^2 - 2x - 4\, dx$

$y = \frac{x^3}{3} - x^2 - 4x + C$ • Plug in x+y to find C

$-6 = \frac{(3)^3}{3} - (3)^2 - 4(3) + C$

$-6 = -12 + C$

$C = 6$

• Plug in C to gen. eq. to get particular solution

$y = \frac{x^3}{3} - x^2 - 4x + 6$

6. $\frac{dx}{y} = \frac{4\,dy}{x}$; $y(4) = -2$

$\int 4y\, dy = \int x\, dx$

$\frac{4y^2}{2} = \frac{x^2}{2} + C$

$2y^2 = \frac{x^2}{2} + C$

$2(-2)^2 = \frac{(4)^2}{2} + C$

$8 = 8 + C$

$C = 0$

$2y^2 = \frac{x^2}{2} + 0$

$\sqrt{y^2} = \sqrt{\frac{x^2}{4}}$

$y = \pm\frac{x}{2}$