Final Exam Review – Meaning of Derivatives and Derivative Rules

1. What is the difference between average rate of change on $[a, b]$ and instantaneous rate of change at $x = a$?

Avg. rate of change is the slope between 2 points + instantaneous rate of change is the slope of the tangent line at 1 point. IROC is a derivative.

Use the limit definition of a derivative to evaluate.

2. $\lim_{h\to 0} \frac{8\left(\frac{1}{2}+h\right)^8 - 8\left(\frac{1}{2}\right)^8}{h}$ $x = \frac{1}{2}$

$f(x) = 8x^8$

$f'(x) = 64x^7$

$f'(\frac{1}{2}) = 64(\frac{1}{2})^7 = 64\left(\frac{1}{128}\right) = \frac{1}{2}$

3. $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right) - \cos\left(\frac{\pi}{2}\right)}{h}$ $x = \frac{\pi}{2}$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f'(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

4. Approximate the derivative using a graph: Figure 5 shows the graph of an object's position $s = s(t)$ on an $s - axis$ as a function of the time t. What is the object's approximate velocity in the positive s –direction at

a. $t = 3$ 0 yds/min.

b. $t = 2$ (1,40) (3,70) $\frac{70-40}{3-1} =$ 15yds/min.

c. $t = 6$ (6,40) (8,22) $\frac{22-40}{8-6} = -9$ yds/min.

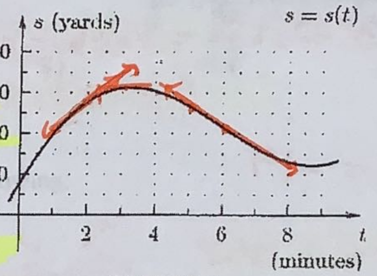


FIGURE 5

*you can use different points

5. Suppose that the amount of water in gallons in a holding tank at t minutes is given by $V(t) = 2t^2 - 16t + 35$. Determine each of the following by finding the derivative of $V(t)$

a. Find the average rate of change on $[1,5]$

$v(1) = 21$ $v(5) = 5$ $\frac{5-21}{5-1} = \frac{-16}{4} = -4$ gal/min.

b. Find the rate of change in the volume of water in the tank at $t = 1$ minute

$v'(t) = 4t - 16$

$v'(1) = 4 - 16 = -12$ gal/min

c. Find the rate of change in the volume of water in the tank at $t = 5$ minutes

$v'(5) = 4(5) - 16$

$= 4$ gal/min.

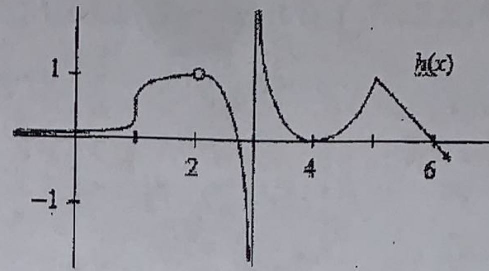
d. Is the volume of water in the tank ever not changing? If so, when? yes

$0 = v'(t)$

$0 = 4t - 16$

$t = 4$ min.

6. Use the graph shown to describe where and why $h(x)$ is not differentiable.



$x=1$ vertical tangent

$x=2$ removable discontinuity

$x=3$ infinite discontinuity

$x=5$ cusp

7. $f(x) = 4x^2 - 3x + 2$. Find the equation of the tangent line and normal line to the curve at the point where $x = -2$.

$f(-2) = 4(-2)^2 - 3(-2) + 2 = 16 + 6 + 2 = 24$

$f'(x) = 8x - 3$

$f'(-2) = 8(-2) - 3 = -19$

Tangent Line: $y - 24 = -19(x+2)$

or $y = -19x - 14$

Normal line: $y - 24 = \frac{1}{19}(x+2)$

$y = \frac{1}{19}x + \frac{458}{19}$

8. The line tangent to the graph of $y = f(x)$ at $x = 3$ passes through the points $(-1,5)$ and $(3,-1)$. Find $f(3)$ and $f'(3)$

$f(3) = -1$ given

$f'(3) = \frac{-1-5}{3--1} = \frac{-6}{4} = \frac{-3}{2}$

$f'(3) = \frac{-3}{2}$

9. Suppose $g(x) = 3x + 1$ is the equation of the tangent line to the graph of $y = f(x)$ at $a = 2$. What is $f(2)$ and $f'(2)$

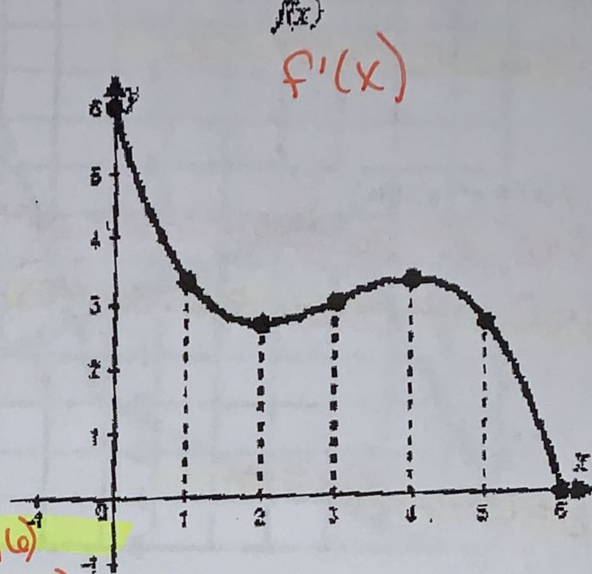
$f(2) = 3(2) + 1$

$f(2) = 7$

$f'(2) = 3$

10. Use the graph at the right to complete the table.

Condition	Domain Interval/Value
$f'(x) < 0$	$(0,2) \cup (4,6)$
$f'(x) = 0$	$x = 2, 4$
$f'(x) > 0$	$(2,4)$
$f''(x) < 0$ conc ↓	$(3,6)$
$f''(x) = 0$ Inf. Pt	$x = 3$
$f''(x) > 0$ conc. ↑	$(0,3)$



At what point is f speeding up? Slowing down?

$f'(x)$ — − + + − − (0, 2, 4, 6)

$f''(x)$ + + + + − − − − (0, 3, 6)

speeding up: $(2,3) \cup (4,6)$

slowing down: $(0,2) \cup (3,4)$

11. Given $f(x) = x^3 - 3x^2 - 9x + 6$, find the following:

a. Where is $f(x)$ increasing? Decreasing?

$f'(x) = 3x^2 - 6x - 9$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$f'(x)$: + (−1) − (3) +

Incr: $(-\infty, -1) \cup (3, \infty)$ bc $f'(x) < 0$

Decr: $(-1, 3)$ bc $f'(x) > 0$

b. Where does $f(x)$ have a local max? Local min?

Local max at $x = -1$

Local min at $x = 3$

c. Where is $f(x)$ concave up? Concave down?

$f''(x) = 6x - 6$

$0 = 6x - 6$

$x = 1$

f'': − (1) +

Conc. up: $(1, \infty)$ bc $f''(x) > 0$

Conc. Down: $(-\infty, 1)$ bc $f''(x) < 0$

12. Find where the following functions have horizontal tangents:

a. $f(x) = 3x^2 - 15$

$f'(x) = 6x$

$0 = 6x$

$x = 0$

b. $f(x) = \sin(2x) - x$

$f'(x) = 2\cos 2x - 1$

$0 = 2\cos(2x) - 1$

$\cos(2x) = \frac{1}{2}$

$2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

c. $f(x) = 12x^2 - 3x^3$

$f'(x) = 24x - 9x^2$

$0 = -3x(3x - 8)$

$x = 0, 8/3$

d. $f(x) = \frac{3}{4}x^4 - \frac{3}{2}x^2$

$f'(x) = 3x^3 - 3x$

$0 = 3x(x^2 - 1)$

$x = -1, 0, 1$

13. Evaluate the derivative using derivative rules

a. $f(x) = (3x^2 - 4)(5x + 3)$

$f(x) = 15x^3 + 9x^2 - 20x - 12$

$f'(x) = 45x^2 + 18x - 20$

b. $y = \frac{6x^2}{\sin x}$

$y' = \frac{\sin x \cdot 12x - 6x^2 \cos x}{\sin^2 x}$

$y' = \frac{12x \sin x - 6x^2 \cos x}{\sin^2 x}$

c. $f(x) = e^{2x}\sin(3x)$

$f'(x) = e^{2x} \cdot 3\cos(3x) + \sin(3x) \cdot 2e^{2x}$

$f'(x) = e^{2x}(3\cos(3x) + 2\sin(3x))$

d. $f(x) = e^{\cot x}$

$f'(x) = e^{\cot x} \cdot -\csc^2 x$

$f'(x) = -\csc^2 x \cdot e^{\cot x}$

e. $y = e^{x^2 - 3x}$

$y = (2x - 3)e^{x^2 - 3x}$

f. $y = \sqrt[3]{x^7} + \frac{1}{\sqrt[5]{x^2}} - \frac{2}{x}$

$y = x^{7/3} + x^{-2/5} - 2x^{-1}$

$y' = \frac{7}{3}x^{4/3} - \frac{2}{5}x^{-7/5} + 2x^{-2}$

$y' = \frac{7}{3}x^{4/3} - \frac{2}{5x^{7/5}} + \frac{2}{x^2}$

g. $g(x) = 5x^4 \ln(2x^3 + 6)$

$g'(x) = 5x^4 \cdot \frac{6x^2}{2x^3 + 6} + 20x^3 \ln(2x^3 + 6)$

$g'(x) = \frac{30x^6}{2x^3 + 6} + 20x^3 \ln(2x^3 + 6)$

h. $f(x) = \tan^{-1}(e^{3x})$

$f'(x) = \frac{1}{1 + (e^{3x})^2} \cdot 3e^{3x}$

$f'(x) = \frac{3e^{3x}}{1 + e^{6x}}$

i. $y = \cos^6(3x - 2)$

$y' = 6\cos^5(3x - 2) \cdot -\sin(3x - 2) \cdot 3$

$y' = -18\cos^5(3x - 2)\sin(3x - 2)$

j. $y = \ln(5x^2 - 3x) + 7\tan(x^3) - \frac{8}{x}$

$y' = \frac{10x - 3}{5x^2 - 3x} + 7 \cdot 3x^2 \sec^2(x^3) + \frac{8}{x^2}$

$y' = \frac{10x - 3}{5x^2 - 3x} + 21x^2 \sec^2(x^3) + \frac{8}{x^2}$

14. Find the derivative using the chart.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-4	6	3	5	6
-2	3	-3	7	1
0	2	2	4	-2
1	-2	-5	-1	3
2	4	1	12	-3
4	5	-4	2	-4

a. Find $\frac{d}{dx}\left[\frac{g(x)}{f(x)}\right]$ when $x = -2$

$\frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2} = \frac{3(1) - (-3)(7)}{3^2}$

$= \frac{24}{9}$

b. $\frac{d}{dx}\left[x^2 - f(\sqrt{x})\right]$ when $x = 4$

$2x - f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

$2(4) - 12 \cdot \frac{1}{2 \cdot 2} = 8 - 3 = 5$

c. Find $\frac{d}{dx}\left[g^{-1}(x)\right]$ when $x = 3$

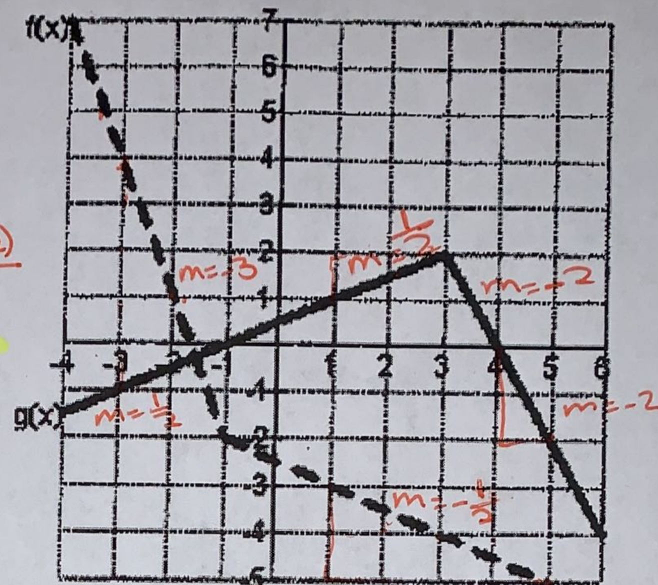
$\frac{d}{dx}\left[g^{-1}(3)\right]$

$\frac{1}{g'(g^{-1}(3))} = \frac{1}{g'(-4)} = \frac{1}{6}$

15. f and g are functions whose graphs are given below. Find the following:

a. $p(x) = f(x) \cdot g(x)$, find $p'(5)$

$p'(5) = f(5)\cdot g'(5) + g(5)\cdot f'(5)$
$= -5 \cdot -2 + -2(-\frac{1}{2})$
$= 10 + 1$
$= 11$



b. $m(x) = \frac{x^2}{f(x)}$, find $m'(-2)$

$m'(-2) = \frac{f(-2)\cdot 2(-2) - (-2)^2 f'(-2)}{[f(-2)]^2}$
$= \frac{1 \cdot -4 - 4(-3)}{(1)^2} = 8$

c. $h(x) = 3g(f(x))$, find $h'(2)$

$h'(2) = 3g'(f(2)) \cdot f'(2)$
$= 3g'(-3\frac{1}{2}) \cdot \frac{-1}{2}$
$= 3(\frac{1}{2})(\frac{-1}{2})$
$= -3/4$

16. If $f(x) = \frac{4}{(1-x)^2}$, find the linearization for $f(x)$ at 0. Use the linearization of $f(x)$ to approximate the value of $f(0.07)$.

$f'(x) = \frac{-8}{(1-x)^3} \cdot -1$ $\qquad f(0) = 4$
$f'(x) = \frac{8}{(1-x)^3}$ $\qquad y - 4 = 8(x - 0)$
$y(x) = 8x + 4$
$f'(0) = 8$ $\qquad y(0.07) = 8(0.07) + 4$ $\qquad y(0.07) = 8.56$

17. The equation of motion of a particle is: $s(t) = t^3 - 15t^2 + 48t + 10$, where s is in meters and t is in seconds. Find the following when $0 < t < 9$. Be sure to include units where appropriate.

a. Find the velocity function in terms of t

$v(t) = 3t^2 - 30t + 48$

b. When is the particle at rest?

$0 = t^2 - 10t + 16$
$0 = (t-8)(t-2)$ $\qquad t = 2, 8$ sec.

c. When is the particle moving forward?

$+ \quad - \quad +$
$0 \quad 2 \quad 8 \quad 9$

$(0,2) \cup (8,9)$ sec.

d. What is the acceleration function in terms of t?

$a(t) = 6t - 30$

e. When is the particle slowing down?

$v(t)$: $+ \quad - \quad +$ ($0 \quad 2 \quad 8 \quad 9$)
$a(t)$: $- \quad - \quad + \quad +$ ($0 \quad 5 \quad 9$)

$(0,2) \cup (8,9)$ sec

f. What is the displacement from $t = 0$ to $t = 9$?

$s(0) = 10$
$s(9) = -44$ $\qquad -44 - 10 = -54$ meters

g. What is the total displacement traveled from $t = 0$ to $t = 9$

$s(2) = 54$
$s(8) = -54$

$s(2) - s(0) = 54 - 10 = 44$m
$|s(8) - s(2)| = |-54 - 54| = 108$m
$s(9) - s(8) = -44 - (-54) = 10$m

162 meters