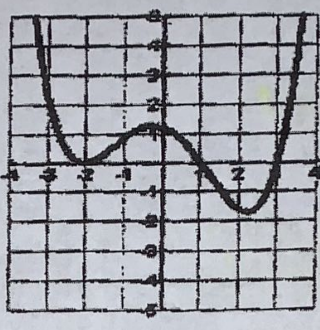
Final Exam Review - Derivative Applications

1. Given the graph of $f'(x)$, find the following intervals or x values where: (estimate to the nearest $\frac{1}{4}$ unit)



a. $f(x)$ is increasing. Justify.

$(-\infty, -2) \cup (-2, 1) \cup (3, 4)$
bc $f'(x)$ is positive

b. $f(x)$ has horizontal tangents. Justify.

$x = -2, 1, 3$ bc $f'(x) = 0$

c. $f(x)$ is concave down. Justify.

$(-\infty, -2) \cup (-1/4, 9/4)$
bc $f'(x)$ is decreasing

d. $f(x)$ has a local minimum. Justify.

$x = 3$ bc $f'(x)$ changes from negative to positive

e. $f(x)$ has a point of inflection. Justify.

POI at $x = -2, -1/4, 9/4$ bc
$f'(x)$ changes b/n increasing + decreasing

2. The radius of a circle is increasing at a rate of 3 cm/sec. How fast is the area of the circle changing when the radius is 5 cm long?

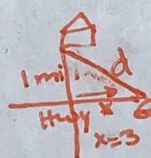
K: $dr/dt = 3$ cm/sec
F: dA/dt
W: $r = 5$ cm

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(5)(3) = 30\,cm^2/sec$

3. A road perpendicular to a highway leads to a farmhouse located 1 mile away. An automobile travels past the farmhouse at a speed of 60 mph. How fast is the distance between the automobile and the farmhouse increasing when the automobile is 3 miles past the intersection of the highway and the road?



$1^2 + 3^2 = d^2$
$d = \sqrt{10}$

K: $\frac{dx}{dt} = 60$ mph
F: $\frac{dd}{dt}$
W: $x = 3$

$x^2 + 1^2 = d^2$

$2x\frac{dx}{dt} = 2d\frac{dd}{dt}$

$\cancel{2}(3)(60) = \cancel{2}\sqrt{10}\frac{dd}{dt}$

$\frac{dd}{dt} = \frac{180}{\sqrt{10}}$ or $18\sqrt{10}$ mph

4. Air is being pumped into a spherical balloon at $5\ cm^3/min$. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

K: $dV/dt = 5\,cm^3/min$
F: dr/dt
W: $d = 20$ cm; $r = 10$ cm

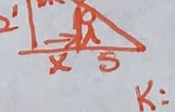
$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$5 = 4\pi(10)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{400\pi} = \frac{1}{80\pi}$ cm/min

5. A light is on the top of a 12 ft tall pole and a 5ft 6in tall person is walking away from the pole at a rate of 2 ft/sec.



a. At what rate is the ~~tip of the~~ shadow moving away from the pole when the person is 25 ft from the pole?

K: $\frac{dx}{dt} = 2$ ft/sec
F: $\frac{dS}{dt}$
W: $x = 25$

$\frac{12}{x+S} = \frac{5.5}{S}$

$5.5x + 5.5S = 12S$

$5.5x = 6.5S$

$5.5\frac{dx}{dt} = 6.5\frac{dS}{dt}$

$6.5\frac{dS}{dt} = 11$

$\frac{dS}{dt} = \frac{11}{6.5}$

$\frac{dS}{dt} = \frac{22}{13}$ ft/sec

b. At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

Tip $= S + x$

$\frac{d\,Tip}{dt} = \frac{dS}{dt} + \frac{dx}{dt}$

$= \frac{22}{13} + 2$

$= \frac{48}{13}$ ft/sec

6. Determine all the numbers c which satisfy the conclusion of the Mean Value Theorem for the following function $f(x) = \frac{1}{4}x^3 + 1$ over the interval $[-2,2]$. $f(x)$ is cont. + diff. on $(-2,2)$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{3}{4}c^2 = \frac{3-(-1)}{2-(-2)}$

$\frac{3}{4}c^2 = 1$

$c^2 = \frac{4}{3}$

$c = \pm\frac{2}{\sqrt{3}}$

7. Find the absolute maximum and the absolute minimum of the function $f(x) = x^3 - x^2 - x + 2$ on the interval $[-10,2]$. Justify your answer.

$f'(x) = 3x^2 - 2x - 1$

$0 = (3x+1)(x-1)$

$x = -1/3, 1$

$f(-10) = -1088$ Abs min

$f(-\frac{1}{3}) = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 2 = \frac{59}{27}$

$f(1) = 1$

$f(2) = 8 - 4 - 2 + 2 = 4$ Abs max

8. Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other is a maximum.

$x = 3$

$y = 6$

$x + y = 9 \rightarrow x = 9 - y$

$x \cdot y^2 = max$

$(9-y)y^2 = max$

$9y^2 - y^3 = max$

$18y - 3y^2 = 0$

$3y(6-y) = 0$

$y = 0, 6$

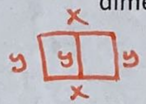
$-$ $+$ $-$

0 6

$y = 6$

$x = 9 - 6$

9. We want to build a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



$2x + 3y = P$

$2x + 3y = 500$

$x = 250 - \frac{3}{2}y$

$x = 250 - \frac{3}{2} \cdot \frac{250}{3}$

$x = 125$

$A = xy$

$A = (250 - \frac{3}{2}y)y$

$A = 250y - \frac{3}{2}y^2$

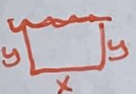
$A' = 250 - 3y$

$0 = 250 - 3y$

$y = \frac{250}{3}$

125 ft x $\frac{250}{3}$ ft

10. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



$2400 = x + 2y$

$x = 2400 - 2y$

$x = 2400 - 2(600)$

$x = 1200'$

$A = xy$

$A = (2400 - 2y)y$

$A = 2400y - 2y^2$

$A' = 2400 - 4y$

$0 = 2400 - 4y$

$y = 600'$

1200 ft x 600 ft

11. A manufacturer determines that x employees on a certain production line will produce y units per month where $y = 75x^2 - 0.2x^4$. To obtain maximum monthly production, how many employees should be assigned to the production line?

$y = 75x^2 - 0.2x^4$

$y' = 150x - .8x^3$

$0 = x(150 - .8x^2)$

$x = 0$ $\quad 150 - .8x^2 = 0$

$-\frac{4}{5}x^2 = -150$

$x^2 = \frac{750}{4} = \frac{375}{2}$

$x = \sqrt{\frac{375}{2}} \approx 13.6$ or 14 people

12. Find the point on the parabola $y = x^2 + 1$ that is closest to the point $(3,1)$ (x, x^2+1)

$d = \sqrt{(x-3)^2 + (x^2+1-1)^2}$

$d = \sqrt{(x-3)^2 + x^4}$

$d' = \frac{1}{2\sqrt{(x-3)^2+x^4}} \cdot 2(x-3) + 4x^3$

$0 = \frac{2(x-3) + 4x^3}{2\sqrt{(x-3)^2+x^4}}$

$0 = 2x - 6 + 4x^3$

$0 = 2x^3 + x - 3$

$$\begin{array}{r|rrrr} 1 & 2 & 0 & 1 & -3 \\ & \downarrow & 2 & 2 & 3 \\ \hline & 2 & 2 & 3 & 0 \end{array}$$

$\frac{-2 \pm \sqrt{4 - 4(2)(3)}}{2}$ = imag.

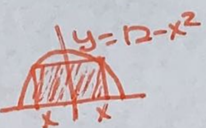
$x = 1$

$y = x^2 + 1$

$y = 2$

$(1, 2)$

13. Find the dimensions of the rectangle with maximum area that has its base on the $x - axis$ and its other two vertices along the $x - axis$ and lying on the parabola $y = 12 - x^2$.



$b = 2x$

$h = 12 - x^2$

$b = 2(2) = 4$

$h = 12 - 2^2 = 8$

$A = bh$

$A = 2x(12 - x^2)$

$A = 24x - 2x^3$

$A' = 24 - 6x^2$

$0 = 24 - 6x^2$

$6x^2 = 24$

$x = \pm 2$

4 x 8 units