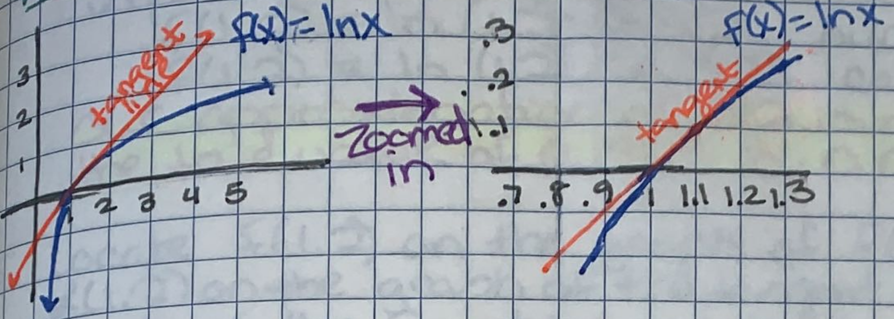
Keeper 5.1 - Linear Approximation

Linear Approximation Visual



If you zoom in really close, it makes your graph look linear. Tangent line almost overlaps so you can use the equation of the tangent line to approx. the curve.

Steps:

1. write the equation of the tangent line to the graph of $f(x)$.
2. Solve the equation for y, then replace y with $L(x)$.
   This equation of the tangent line is called linear approximation.
3. Use the linearization $\mathcal{L}(x)$ to approximate the given value.

Linearization Line (Tangent Line): $\mathcal{L}(x) = f'(a)(x-a) + f(a)$

a. Write the equation of the line tangent to the graph of $f(x) = \ln x$ at $x=1$

$f'(x) = \frac{1}{x}$ $\quad$ $f(x) = \ln x$ $\quad$ $y - 0 = 1(x-1)$

$f'(1) = 1$ $\quad$ $f(1) = \ln 1 = 0$

b. Solve the equation in part a for y, then replace with $L(x)$. This eq. of the tangent line is called linear approximation.

$y - 0 = 1(x-1)$

$y = x - 1$

$\mathcal{L}(x) = x - 1$

c. Use the linearization $\mathcal{L}(x)$ to approximate $f(0.95)$

$f(0.95) \approx \mathcal{L}(0.95)$

$\mathcal{L}(x) = x - 1$

$\mathcal{L}(0.95) = 0.95 - 1$

$\mathcal{L}(0.95) = -0.05$

$\therefore f(0.95) \approx -0.05$

d. Use the linearization $\mathcal{L}(x)$ to approximate $f(1.2)$

$f(1.2) \approx \mathcal{L}(1.2)$

$\mathcal{L}(x) = x - 1$

$\mathcal{L}(1.2) = 1.2 - 1$

$\mathcal{L}(1.2) = 0.2$

$\therefore f(1.2) \approx 0.2$

Since the initial quest. asks for $x=1$, only choose numbers close to 1 (2 would be too far)

cont. 5.1

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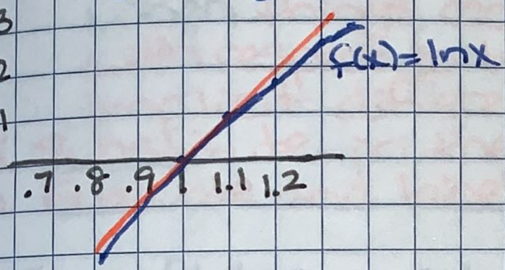
e. Find $f(1.2)$ and compare it to the linear approximation in part d.

$f(1.2) = \ln(1.2) \approx 0.2$ our approximation

$f(1.2) = \ln(1.2) = 0.1823$ calculator answer

Our approximation of 0.2 is an overestimate of the true value of 0.1823 by 0.0177.

f. Locate $f(1.2)$ on the graph of $f(x) = \ln x$ and locate $\mathcal{L}(1.2)$ on the graph of the tangent line. Looking at the graph of the function and its tangent, can you explain whether your estimate is an over or underestimate? Why might this be?



The tangent line is above the graph of f(x) therefore our approximation is an overestimate. Additionally, the graph is concave down at x=1.2 thus the tangent line would have to be above it.

5.2 Related Rates

Related Rates Organization:

1. Read the problem
2. Answer these questions:
   1. Know: what rate are you given? ie dv/dt
   2. Find: what rate are you asked to find?
   3. When: Most of the time you are given a time when you want to find the rate. This info is only used after the derivative is taken. If it isn't given, you don't need it.
3. Write an equation that relates the Know rate with the Find rate.
4. Take a derivative. Since we are concerned about when things occur we will be taking the derivative with respect to time (t) and we will need to do implicit differentiation.
5. Substitute in the Known rate and the when time.
6. Evaluate and label your answer.

1. Joe inflates a spherical balloon. Air is entering at a rate of 15 cm³/sec. How fast is the radius changing when the radius is 10 cm?

K: $dv/dt = 15\text{cm}^3/\text{sec}$

F: dr/dt

W: $r = 10\text{ cm}$

$$V = \tfrac{4}{3}\pi r^3$$
$$dv/dt = 4\pi r^2\, dr/dt$$
$$15 = 4\pi(10)^2\, dr/dt$$
$$\frac{dr}{dt} = \frac{15}{400\pi}$$
$$dr/dt = \frac{3}{80\pi}\text{ cm/sec}$$

2. A piece of ice cut in the shape of a cube melts uniformly so that its volume decreases at 3 cm³/sec. How fast is the surface area decreasing when the edge of the cube is 5 cm?

K: $dv/dt = -3\text{cm}^3/\text{sec}$

F: ds/dt

W: $e = 5\text{cm}$ so $V = 125\text{cm}^3$

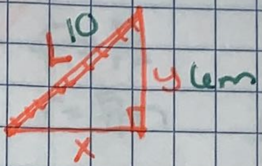
$$V = e^3$$
$$\sqrt[3]{V} = e$$
$$S = 6e^2$$
$$S = 6\sqrt[3]{V}^2$$

Substitute e into SA formula

$$ds/dt = 4\, v^{-1/3}\, dv/dt$$
$$ds/dt = 4(125)^{-1/3}(-3)$$
$$ds/dt = \frac{-12}{\sqrt[3]{125}}$$
$$ds/dt = \frac{-12}{5}\text{ cm}^2/\text{sec}$$

3. A ladder 10 meters long is leaning against a vertical wall with its other end on the ground. The top end of the ladder is sliding down the wall. When the top end is 6 meters from the ground it is sliding down at 2m/sec. How fast is the bottom moving away from the wall at this instant?

K: $dy/dt = -2$ m/s
F: dx/dt
W: $x=8$ $y=6$ $L=10$



$$x^2+y^2=10^2$$
$$2x\,dx/dt+2y\,dy/dt=0$$
$$2(8)dx/dt+2(6)(-2)=0$$
$$dx/dt=24/16$$
$$\frac{dx}{dt}=3/2 \text{ m/s}$$

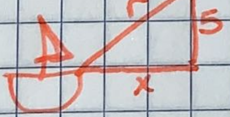
4. A right circular cylinder of a constant volume is being flattened. At the moment when its radius is 3cm, the height is 4cm & the height is decreased at the rate of 0.2 cm/sec. At that moment, what is the rate of change of the radius?

K: $dh/dt = -.2$ cm/sec
F: dr/dt
W: $r=3$cm $h=4$cm

$$V=\pi r^2h$$
$$dV/dt=\pi r^2\,dh/dt+h\cdot 2\pi r\,dr/dt$$
$$0=\pi(3)^2(-.2)+2\pi(3)(4)\,dr/dt$$
$$0=-1.8\pi+24\pi\,dr/dt$$
$$dr/dt=\frac{-1.8\pi}{-24\pi}$$
$$dr/dt=\frac{3}{40} \text{ cm/sec}$$

5. A boat is pulled into a dock by a rope attached to it + passing through a pulley on the dock positioned 5 meters higher than the boat. If the rope is being pulled in at a rate of 2m/sec, how fast is the boat approaching the dock when it is 12 meters away from the dock?

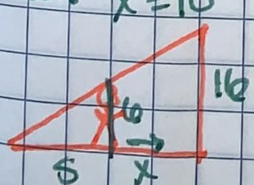
K: $dr/dt = -2$ m/s
F: dx/dt
W: $x=12$ $y=5$
$r=13$



$$x^2+y^2=r^2$$
$$2x\,dx/dt+2y\,dy/dt=2r\,dr/dt$$
$$2(12)dx/dt+2(5)(0)=2(13)(-2)$$
$$24\,dx/dt=-52$$
$$dx/dt=-13/6 \text{ m/s}$$

6. A man 6ft tall walks at a rate of 5ft/sec. toward a street light that is 16ft above the ground. At what rate is the tip of the shadow moving? At what rate is the length of his shadow changing when he is 10ft from the base of the light?

K: $dx/dt = -5$ft/sec
F: $ds/dt + dtip/dt$
W: $x=10$



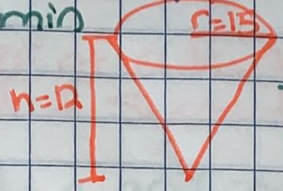
$$\frac{16}{x+s}=\frac{6}{s}$$
$$16s=6x+6s$$
$$10s=6x$$
$$10\,ds/dt=6\,dx/dt$$
$$10\,ds/dt=6(-5)$$
$$ds/dt=-3 \text{ ft/sec}$$

$$Tip=x+s$$
$$dtip/dt=dx/dt+ds/dt$$
$$dtip/dt=-5+-3$$
$$\frac{dtip}{dt}=-8 \text{ ft/sec}$$

Cont. 5.2

7. A water tank has the shape of an inverted right-circular cone, with radius at the top 15 meters & depth 12 meters. Water is flowing into the tank at the rate of 2 cubic meters per minute. How fast is the depth of water in the tank increasing when the depth is 8 meters?

K: $dV/dt = 2 m^3/min$
F: dh/dt
W: $h = 8$



$\frac{r}{h} = \frac{15}{12}$

$r = \frac{5}{4}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{5}{4}h\right)^2 h$

$V = \frac{25}{48}\pi h^3$

$\frac{dV}{dt} = \frac{25}{16}\pi h^2 \frac{dh}{dt}$

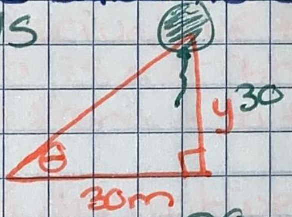
$2 = \frac{25}{16}\pi(8)^2\frac{dh}{dt}$

$2 = 100\pi\, dh/dt$

$\frac{dh}{dt} = \frac{1}{50\pi}$ m/min

8. A balloon rises at a rate of 3 meters per second from a point on the ground 30 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 30 meters above the ground.

K: $dy/dt = 3 m/s$
F: $d\theta/dt$
W: $y = 30$



$\tan\theta = \frac{30}{30}$

$\tan\theta = 1$

$\theta = \pi/4$

$\tan\theta = \frac{y}{30}$

$\sec^2\theta\, d\theta/dt = \frac{1}{30}\, dy/dt$

$(\sqrt{2})^2 d\theta/dt = \frac{1}{30}(3)$

$2\, d\theta/dt = \frac{1}{10}$

$\frac{d\theta}{dt} = \frac{1}{20}$ rad/sec.

5.3 Maximum & Minimum Values

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Local Extrema: A function has a local maximum (or relative max) at c if $f(c) \geq f(x)$ for all x in the neighborhood around c. The function has a local minimum (or relative minimum) at c if $f(c) \leq f(x)$ for all x in the neighborhood around c.

A critical number of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ DNE.

Ways to verify a critical # is a local (relative) max:

1. use the first derivative test:
   Suppose that c is a critical # of a continuous function f
   If f' changes from + or - at c, then f has a local max at c.
2. use the second derivative test:
   If $f'(c) = 0$ & $f''(c) < 0$ (concave down), the f has a local max at c.

Ways to verify a critical # is a local (relative) minimum:

1. use the first derivative test:
   Suppose that c is a critical # of a continuous function f
   If f' changes from - to + at c, then f has a local min. at c.
2. Use the second derivative test:
   If $f'(c) = 0$ & $f''(c) > 0$ (concave up), then f has local min

Examples:

1. Find + classify all critical numbers for $f(x) = x^3 - 3x^2 - 4$

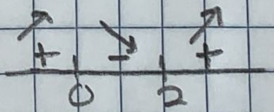
$f(x)$ is continuous

$f'(x) = 3x^2 - 6x$

$0 = 3x^3 - 6x$

$0 = 3x(x-2)$

$x = 0 \quad x = 2$



$x = 0$ local max

$x = 2$ local min

or 2nd deriv. test

$f''(x) = 6x - 6$

* $f''(0) = -6$ ∴ at $x = 0$, $f(x)$ is concave down (max)
* $f''(2) = 6$ ∴ at $x = 2$ $f(x)$ is conc. up (min)

Absolute Extrema

- A function has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in the domain of f.
- A function has an absolute minimum (or global minimum) at c if $f(c) \leq f(x)$ for all x in the domain of f.

cont. 5.3

The Extreme Value Theorem

If f is continuous on a closed interval [a,b] then f attains an absolute max value f(c) & an absolute min value f(d) at some numbers c & d in [a,b]

Steps to finding absolute extrema:

a. Verify the function is continuous & on a closed interval.
b. Find all critical #s of f in (a,b)
c. Find the value of f(x) all at critical #s & at both endpoints
d. The largest y-value is declared the absolute maximum & the smallest y-value is declared the absolute minimum.

Find the absolute maximum & minimum values of $f(x) = x^4 - 2x^3 + 3$ on $[-2,3]$.

f(x) is continuous on $[-2,3]$

$f'(x) = 4x^3 - 6x^2$

$0 = 2x^2(2x-3)$

$x = 0 \quad x = 3/2$

$f(-2) = (-2)^4 - 2(-2)^3 + 3 = 35$ Absolute max is 35

$f(0) = (0)^4 - 2(0)^3 + 3 = 3$

$f(3/2) = (3/2)^4 - 2(3/2)^3 + 3 = 1.3125$ Absolute min is 1.3125

$f(3) = (3)^4 - 2(3)^3 + 3 = 30$

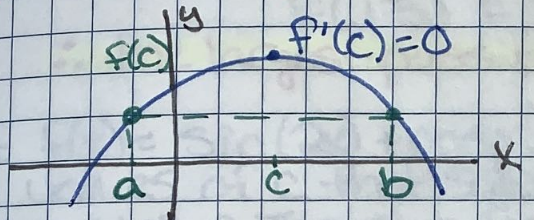
5.4 The Mean Value Theorem

Rolle's Theorem

Suppose $f(x)$ is a function that satisfies all of the following:

1. $f(x)$ is continuous on the closed interval $[a,b]$
2. $f(x)$ is differentiable on the open interval (a,b)
3. $f(a) = f(b)$

Then there is such a # c such that $a<c<b$ and $f'(c)=0$.
In other words, $f(x)$ has a critical point in (a,b)



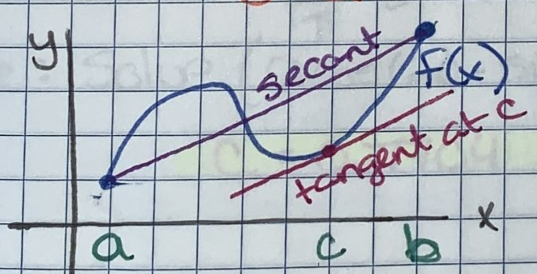
Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following:

1. $f(x)$ is continuous on a closed interval $[a,b]$
2. $f(x)$ is differentiable on the open interval (a,b)

Then there is a # c such that $a<c<b$ and

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{or} \quad f(b)-f(a) = f'(c)(b-a)$$



Examples

1. Determine all the numbers c which satisfy the conclusion of the Mean Value Theorem for $f(x) = x^3+2x^2-x$ on $[-1,2]$

$f(x)$ is continuous on $[-1,2]$
$f(x)$ is differentiable on $(-1,2)$
$\therefore \exists\, c$ s.t. $-1<c<2$

$$f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$$

$$3c^2+4c-1 = \frac{14-2}{3}$$

$$3c^2+4c-5=0$$

not factorable

$$\frac{-4 \pm \sqrt{16-4(3)(-5)}}{6}$$

$$\frac{-4 \pm \sqrt{76}}{6} = \frac{-4 \pm 2\sqrt{19}}{6}$$

$$= \frac{-2 \pm \sqrt{19}}{3}$$

$$c = \frac{-2+\sqrt{19}}{3}$$

cont. 5.4

2. Suppose that we know that f(x) is continuous + differentiable on [6, 15]. Let's also assume that we know that f(6) = -2 + that f'(x) ≤ 10. What is the largest possible value for f(15)?

$$\frac{f(15)-f(6)}{15-6} \leq 10$$

$$\frac{f(15)+2}{9} \leq 10$$

$$f(15) \leq 88$$

∴ the largest possible value of f(15) = 88

3. Let f(x) = sin(2x) + cosx. Use your calculator to find all values of c that satisfy the conclusion of the mean Value Theorem for 0 ≤ x ≤ π.

f(x) is continuous on [0, π]

f(x) is differentiable on (0, π)

$$\frac{f(\pi)-f(0)}{\pi-0} = 2\cos(2c)-\sin c$$

$$\frac{-1-1}{\pi} = 2\cos(2c)-\sin c$$

$$\frac{-2}{\pi} = 2\cos(2c)-\sin c$$

★calc: Solve $\left(2\cos(2x)-\sin(x) = \frac{-2}{\pi}, x\right) \mid 0 \leq x \leq \pi$

c = 0.7704, 2.3712

5.5 Optimization

Steps to Optimization Problems

1. Understand the problem: What is Known? Unknown? What do you want to find?
2. Draw a diagram to visualize + see if things make sense.
3. Introduce notation - label + define variables.
4. Write equations in terms of one variable - Rewrite everything in terms of 1 variable.
5. Find relationships between the variables to simplify into one.
6. Differentiate to find the maximums/minimums.

1. Find 2 positive #s such that their product is 192 and the sum of the first plus three times the second is a minimum.

$x \cdot y = 192$

$x = \frac{192}{y}$

$x + 3y$

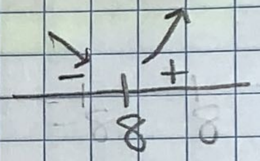
$\frac{192}{y} + 3y$ ← take deriv. + set = 0

$-\frac{192}{y^2} + 3 = 0$

$-192 + 3y^2 = 0$

$y^2 = 64$

$y = \pm 8$ but must be positive

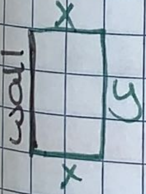


$x = \frac{192}{8}$

$x = 24$

$x = 24$ + $y = 8$

2. A gardener wants to make a rectangular enclosure using a wall as one side + 120m of fencing for the other three sides. Express the area in terms of x, + find the value of x that gives the greatest area.



$P = 2x + y = 120$

$y = 120 - 2x$

$A = xy$

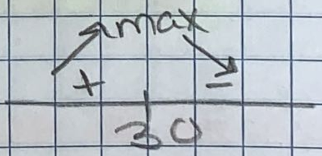
$A = x(120 - 2x)$

$A = 120x - 2x^2$

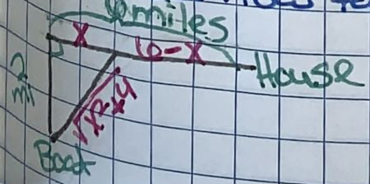
$A' = 120 - 4x$

$0 = 120 - 4x$

$x = 30m$



3. A person in a rowboat two miles from the nearest point on a straight shoreline wishes to reach a house six miles farther down the shore. If the person can row at a rate of 3mi/hr + walk at a rate of 5mi/hr, find the least amount of time required to reach the house. How far from the house should the person land the rowboat?



$D = rt$

$t = \frac{D}{r}$

$T_{total} = T_{row} + T_{walk}$

$T = \frac{\sqrt{x^2+4}}{3} + \frac{6-x}{5}$

★ Calc: Solve$(\frac{d}{dx}(T(x)) = 0, x)$

$x = \frac{3}{2}$ or 1.5 miles

T(1.5) in calc

$T(1.5) = 1.733$ hours

5.6 Interpreting Graphs

Critical Values - y-values where $f'(x) = 0$ or DNE in the domain of $f(x)$

Ex: Find the critical values of $f(x) = x^3 + 6x^2 + 9x + 1$

$$f'(x) = 3x^2 + 12x + 9$$
$$0 = x^2 + 4x + 3$$
$$0 = (x+1)(x+3)$$
$$x = -1, -3$$

Intervals of Increase + Decrease

- Int. of Increase - intervals where $f'(x)$ is positive
- Int of Decrease - intervals where $f'(x)$ is negative

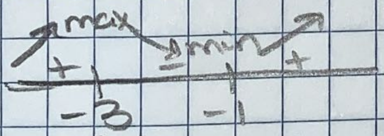
Ex: Find the intervals of increase/decrease of $f(x) = x^3 + 6x^2 + 9x + 1$

Inc $(-\infty, -3) \cup (-1, \infty)$
Decr $(-3, -1)$

Local Extrema: max + min

- local max - where $f'(x)$ changes from + to -
- local min - where $f'(x)$ changes from - to +
- ★ local max/min values are y-values. If asked "where" is local max/min then give x-value.

Ex: Find all local extrema. State what type. $f(x) = x^3 + 6x^2 + 9x + 1$



$f(-3) = 1 \rightarrow$ local max
$f(-1) = -3 \rightarrow$ local min

★ plug into $f(x)$ to get y-value

Absolute max/min - compare the y-values of the endpoints and critical #s

Ex: Find the absolute extrema from $(-3.5, 0)$ for same fxn.

$$f(-3.5) = (-3.5)^3 + 6(-3.5)^2 + 9(-3.5) + 1 = 1/8$$
$$f(-3) = (-3)^3 + 6(-3)^2 + 9(-3) + 1 = 1$$
$$f(-1) = (-1)^3 + 6(-1)^2 + 9(-1) + 1 = -3$$
$$f(0) = 0^3 + 6(0)^2 + 9(0) + 1 = 1$$

Abs. Max = 1
Abs. Min = -3

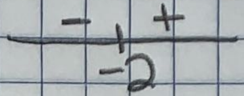
cont. 5.6

Concavity

- concave up occurs where $f''(x)$ is positive
- concave down occurs where $f''(x)$ is negative.

Ex: Find where $f(x)=x^3+6x^2+9x+1$ is concave up + down.

$$f'(x)=3x^2+12x+9$$
$$f''(x)=6x+12$$
$$0=6x+12$$
$$x=-2$$



concave down $(-\infty,-2)$
concave up $(-2,\infty)$

Points of Inflection

A possible pt. of inflection occurs where $f''(x)=0$ or DNE in the domain of $f(x)$.

The actual pt. of inflection occurs where $f''(x)$ changes signs.

Ex: Find all points of inflection $f(x)=x^3+6x^2+9x+1$

$$f''(x)=0$$
$$x=-2$$
$$f(-2)=(-2)^3+6(-2)^2+9(-2)+1$$
$$f(-2)=-1$$

POI: $(-2,-1)$