Derivative Applications

Fall 2020

Day	Date	Topic	Assignment
1	Monday, October 5th	**Keeper 5.1 - Linear Approximation**	Linear Approximation (Packet p. 1 – 2)
2	Tuesday, October 6th	**Keeper 5.2 - Related Rates**	Related Rates – Cubes, Circles, Spheres, and Squares (Packet p. 3-4) Related Rates – Ladders, Cars, Boats (Packet p. 5-6)
3	Wednesday, October 7th	**Optional Q&A Session at 10am** **Review Keeper 5.1-5.2**	Get caught up on all Keeper Notes and Homework
4	Thursday, October 8th	**Keeper 5.2 - Related Rates Continued**	Related Rates – Moving Particles, Angles, and Formulas (Packet p. 7 – 8) Related Rates - Shadows, Cones, Coffee Pots, and Trough (Packet p. 9 – 10)
5	Friday, October 9th	**Keeper 5.3 - Max and Min Values** **Keeper 5.4 – The Mean Value Theorem**	The Extreme Value Theorem and Max/Min Values (Packet p. 11) The Mean Value Theorem (Packet p. 11)
6	Monday, October 12th	**Keeper 5.5 – Optimization**	Optimization Problems (Packet p. 12 – 14)
7	Tuesday, October 13th	**Keeper 5.5 – Optimization Continued**	Optimization Problems – Time Problems (Packet p. 15)
8	Wednesday, October 14th	**Optional Q&A Session at 10am** **Unit 5 Mini Review**	Curve Sketching Review (Packet p. 16 – 18)
9	Thursday, October 15th	**Keeper 5.6 – Interpreting Graphs**	Curve Sketching Practice (Packet p. 19 – 20)
10	Friday, October 16th	**Keeper 5.6 – Continued (Analyzing Graphs)**	Curve Sketching Practice (Packet p. 21 – 22)
11	Monday, October 19th	**Review Derivative Applications**	Derivation App Practice Test (Packet p. 23 – 27)
12	Tuesday, October 20th	**Unit 5 Test - Derivative Applications**	Midterm Review **Midterm on Thursday 10/22**

Linear Approximation

1. If $f(x) = x^3 + 3x$, approximate $f(2.01)$ using linearization.

$f'(x) = 3x^2 + 3$
$f'(2) = 3(2)^2 + 3 = 15$
$f(2) = (2)^3 + 3(2) = 14$
$y - 14 = 15(x - 2)$

$L(x) = 15x - 16$
$L(2.01) = 15(2.01) - 16$
$L(2.01) = 14.15$

2. For the function f, $f' = 2x + 1$ and $f(1) = 4$. What is the approximation for $f(1.2)$ using the tangent line approximation?

$f'(1) = 2(1) + 1$
$f'(1) = 3$
$y - 4 = 3(x - 1)$
$y = 3x + 1$

$L(x) = 3x + 1$
$L(1.2) = 3(1.2) + 1$
$L(1.2) = 4.6$

3. Approximate $\sqrt{24.9} + (24.9)^2$ using linearization.

$f(x) = \sqrt{x} + x^2$
Let $x = 25$
$f(25) = \sqrt{25} + (25)^2 = 630$
$f'(x) = \frac{1}{2\sqrt{x}} + 2x$
$f'(25) = \frac{1}{2\sqrt{25}} + 2(25) = \frac{1}{10} + 50$
$f'(25) = \frac{501}{10}$

$y - 630 = \frac{501}{10}(x - 25)$
$y = \frac{501}{10}x - \frac{2505}{2} + 630$
$L(x) = \frac{501}{10}x - \frac{1245}{2}$
$L(24.9) = \frac{501}{10}(24.9) - \frac{1245}{2}$
$L(24.9) = 624.99$

4. Find an approximate value for $f(-3.9)$ on $f(x) = \sqrt{x^2 + 9}$ using linearization.

$f(-4) = \sqrt{(-4)^2 + 9} = 5$
$f'(x) = \frac{1}{2}(x^2 + 9)^{-1/2} \cdot 2x$
$f'(x) = \frac{x}{\sqrt{x^2 + 9}}$
$f'(-4) = \frac{-4}{\sqrt{(-4)^2 + 9}} = -\frac{4}{5}$
$y - 5 = -4/5\,(x + 4)$

$L(x) = -4/5x - 16/5 + 5$
$L(x) = -4/5x + 9/5$
$L(-3.9) = -4/5(-3.9) + 9/5$
$L(-3.9) = 4.92$

5. Approximate using tangent line approximation: $\sqrt[4]{17}$.

$f(x) = \sqrt[4]{x}$ Let $x = 16$
$f(16) = \sqrt[4]{16} = 2$
$f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}$
$f'(16) = \frac{1}{4\sqrt[4]{16^3}} = \frac{1}{32}$
$y - 2 = \frac{1}{32}(x - 16)$

$L(x) = \frac{1}{32}x + \frac{3}{2}$
$L(17) = \frac{1}{32}(17) + \frac{3}{2}$
$= \frac{17}{32} + \frac{48}{32}$
$L(17) = \frac{65}{32}$

6. Approximate using a tangent line approximation $(8.4)^{\frac{4}{3}}$.

$f(x) = x^{4/3}$ Let $x=8$

$f(8) = \sqrt[3]{8^4} = 16$

$f'(x) = \frac{4}{3}x^{1/3}$

$f'(8) = \frac{4\sqrt[3]{8}}{3} = \frac{8}{3}$

$y - 16 = \frac{8}{3}(x-8)$

$y = \frac{8}{3}x - \frac{64}{3} + 16$

$y(x) = \frac{8}{3}x - \frac{16}{3}$

$y(8.4) = \frac{8}{3}(8.4) - \frac{16}{3}$

$y(8.4) = \frac{256}{15}$

7. Let f be the function given by $f(x) = \frac{2x-5}{x^2-4}$.

a. Find the domain of f

$x^2-4 \neq 0$

$x \neq \pm 2$ $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

b. Write an equation for each vertical and each horizontal asymptote for the graph of f. Justify your answer using calculus.

V.A. $x = -2, 2$

$\lim_{x\to -2^-} f(x) = -\infty$ $\lim_{x\to -2^+} f(x) = \infty$

$\lim_{x\to 2^-} f(x) = \infty$ $\lim_{x\to 2^+} f(x) = -\infty$

HA: $y=0$

$\lim_{x\to -\infty} f(x) = 0$ & $\lim_{x\to \infty} f(x) = 0$

c. Find $f'(x)$ and simplify.

$f'(x) = \frac{(x^2-4)2 - 2x(2x-5)}{(x^2-4)^2} = \frac{2x^2-8-4x^2+10x}{(x^2-4)^2} = \frac{-2x^2+10x-8}{(x^2-4)^2}$

d. Write and equation for the line tangent to f at the point $(0, f(0))$.

$f(0) = 5/4$

$f'(0) = \frac{-8}{16} = -\frac{1}{2}$

$y - 5/4 = -1/2(x-0)$

$y = -1/2x + 5/4$

e. Evaluate $\lim_{x\to 2^+} f(x)$.

$\lim_{x\to 2^+} f(x) = \frac{\#}{0}$ so $<\begin{matrix}\infty \\ \text{or} \\ -\infty\end{matrix}$

$\lim_{x\to 2^+} f(x) = -\infty$

8. If $f(x) = \frac{1}{x^2+1}$ and $g(x) = \sqrt{x}$, find the derivative of $f(g(x))$. Simplify your answer.

$f(x) = (x^2+1)^{-1}$

$f'(g(x))\cdot g'(x)$

$\frac{-1}{(x^2+1)^2} \cdot 2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

$\frac{-1}{(x^2+1)^2}$

$f'(x) = \frac{-1}{(x^2+1)^2}$ $f'(g(x)) = \frac{-2x}{(x^2+1)^2}$

$g'(x) = \frac{1}{2\sqrt{x}}$

9. Evaluate the limit: $\lim_{x\to 2} \frac{(\sqrt{x^2-2}-\sqrt{-x+4})}{x-2}\frac{(\sqrt{x^2-2}+\sqrt{-x+4})}{(\sqrt{x^2-2}+\sqrt{-x+4})}$

$\lim_{x\to 2} \frac{x^2-2+x-4}{(x-2)(\sqrt{x^2-2}+\sqrt{-x+4})}$

$\lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x^2-2}+\sqrt{-x+4})}$

$\lim_{x\to 2} \frac{2+3}{\sqrt{4-2}+\sqrt{-2+4}} = \frac{5}{\sqrt{2}+\sqrt{2}}$

$= \frac{5}{2\sqrt{2}}$

Related Rates – Cubes, Circles, Spheres, and Squares

1. All edges of a cube are expanding at a rate of 3 cm/sec. How fast is the volume changing when each edge is 1 cm?

K: $de/dt = 3cm/sec$
F: dV/dt
W: $e = 1\,cm$

$V = e^3$
$dV/dt = 3e^2\, de/dt$
$dV/dt = 3(1)^2(3)$
$dV/dt = 9\, cm^3/sec$

2. The volume of a cube is decreasing at a rate of 12 cubic meters per hour. How fast is the total surface area decreasing when the surface area is 24 m^2?

K: $dV/dt = -12m^3/hr$
F: dS/dt
W: $SA = 24m^2$
$24 = 6e^2$
$e = 2$ so $V = 8$

$S = 6e^2$ $\quad V = e^3$
$S = 6\sqrt[3]{V^2}$ $\quad e = \sqrt[3]{V}$
$dS/dt = 4V^{-1/3}\, dV/dt$
$dS/dt = 4 \frac{1}{\sqrt[3]{8}} \cdot -12 = -24m^2/hr$

3. The radius of a circle is increasing at the rate of 5 in/min. At what rate is the area increasing when the radius is 10 inches?

K: $dr/dt = 5in/min$
F: $dA/dt =$
W: $r = 10in$

$A = \pi r^2$
$dA/dt = 2\pi r\, dr/dt$
$dA/dt = 2\pi(10)(5)$
$dA/dt = 100\pi\, in^2/min$

4. A stone in a still pond creates a circular ripple whose radius increases at a constant rate of 3 ft/s. At what rate is the area enclosed by the ripple increasing 8 s after the stone strikes the pond?

K: $dr/dt = 3ft/sec$
F: dA/dt
W: $t = 8$ so $r = 24ft$

$A = \pi r^2$
$dA/dt = 2\pi r\, dr/dt$
$dA/dt = 2\pi(24)(3)$
$dA/dt = 144\pi\, ft^2/sec$

5. A pebble is dropped into a calm pond creating ripples whose radius increases at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area of the disturbed water changing?

K: $dr/dt = 1ft/sec$
F: dA/dt
W: $r = 4ft$

$A = \pi r^2$
$dA/dt = 2\pi r\, dr/dt$
$dA/dt = 2\pi(4)(1)$
$dA/dt = 8\pi\, ft^2/sec$

6. The radius of a sphere is increasing at a constant rate of 0.05 cm/sec. At the time when the radius of the sphere is 10 cm, what is the rate of increase of the volume?

K: $dr/dt = 0.05 \text{ cm/sec}$
F: dV/dt
W: $r = 10\text{cm}$

$V = \frac{4}{3}\pi r^3$
$dV/dt = 4\pi r^2 \, dr/dt$
$dV/dt = 4\pi (10)^2 (0.05)$
$dV/dt = 20\pi \text{ cm}^3/\text{sec}$

7. A spherical balloon is inflated at the rate of four cubic feet per minute. At what rate is the radius changing when $r = 24 \; in$?

K: $dV/dt = 4 \text{ ft}^3/\text{min}$
F: dr/dt
W: $r = 24\text{in}$ or 2ft

$V = \frac{4}{3}\pi r^3$
$dV/dt = 4\pi r^2 \, dr/dt$
$4 = 4\pi (2)^2 \, dr/dt$
$\frac{dr}{dt} = \frac{1}{4\pi} \text{ ft/sec}$

8. Air is being pumped into a spherical balloon at the rate of 4.5 cubic inches per minute. Find the rate of change of the radius when the radius is 2 inches.

K: $dV/dt = 4.5 \text{ in}^3/\text{min}$
F: dr/dt
W: $r = 2\text{in}$

$V = \frac{4}{3}\pi r^3$
$dV/dt = 4\pi r^2 \, dr/dt$
$4.5 = 4\pi (2)^2 \, dr/dt$
$\frac{dr}{dt} = \frac{4.5}{16\pi} = \frac{45}{160\pi} = \frac{9}{32\pi} \text{ in/min}$

9. How fast is the area of a square increasing when the side is 3 m in length and growing at a rate of 0.8 m/min?

K: $ds/dt = 0.8 \text{m/min}$
F: dA/dt
W: $s = 3\text{m}$

$A = s^2$
$dA/dt = 2s \, ds/dt$
$dA/dt = 2(3)(.8)$
$dA/dt = 4.8 \text{ m}^2/\text{min}$

10. A rectangle has a fixed area of 100 $unit^2$. Its length is increasing at 2 units/sec. Find the length at the instant the width is decreasing at 0.5 units/sec.

K: $dL/dt = 2 \text{ u/sec}$
F: L
W: $dw/dt = -0.5 \text{ unit/sec}$

$A = LW$
$100 = LW$
$W = \frac{100}{L}$

$A = L \cdot W$
$\frac{dA}{dt} = L\frac{dw}{dt} + W\frac{dL}{dt}$
$0 = L(-\frac{1}{2}) + \frac{100}{L}(2)$
$\frac{L}{2} = \frac{200}{L}$
$L^2 = 400$
$L = 20 \text{ units}$

11. A screen saver displays the outline of a 3 cm by 2 cm rectangle and then expands the rectangle in such a way that the 2 cm side is expanding at the rate of 4 cm/sec and the proportions of the rectangle never change. How fast is the area of the rectangle increasing when its dimensions are 12 cm by 8 cm?

K: $dw/dt = 4\text{cm/s}$ $dL/dt = 6\text{cm/s}$
F: dA/dt
W: $L = 12\text{cm}$ $W = 8\text{cm}$

w 2
3 L

$\frac{2}{3} = \frac{w}{L}$
$3w = 2L$
$L = \frac{3}{2}w$

$\frac{2}{3} = \frac{4}{L}$
$12 = 2L$

$A = LW$
$\frac{dA}{dt} = L\frac{dw}{dt} + W\frac{dL}{dt}$
$\frac{dA}{dt} = 12(4) + 8(6)$
$dA/dt = 96 \text{ cm}^2/\text{sec}$

Related Rates – Ladders, Cars, Boats, etc.

1. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the lop of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

K: $dx/dt = 1$ ft/sec
F: dy/dt
W: $x = 6'$ $y = 8'$ $L = 10'$

$$x^2 + y^2 = L^2$$
$$2x\,dx/dt + 2y\,dy/dt = 2L\,dL/dt$$
$$2(6)(1) + 2(8)\,dy/dt = 2(10)(0)$$
$$dy/dt = \frac{-12}{16}$$
$$dy/dt = \frac{-3}{4} \text{ ft/sec}$$

2. A ladder leans against a wall with the bottom of the ladder 8 feet from the wall. The top of the ladder slips down the wall at a rate of 4 ft/sec while the bottom of the ladder is being pulled away at a rate of 3 ft/sec. How long is the ladder?

K: $\frac{dy}{dt} = -4$ ft/sec $\frac{dx}{dt} = 3$ ft/sec
F: L
W: $x = 8$ ft

$$x^2 + y^2 = L^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2L\frac{dL}{dt}$$
$$2(8)(3) + 2y(-4) = 2L(0)$$
$$-8y = -48$$
$$y = 6 \text{ ft}$$

$$x^2 + y^2 = L^2$$
$$8^2 + 6^2 = L^2$$
$$64 + 36 = L^2$$
$$100 = L^2$$
$$L = 10 \text{ ft}$$

3. If one leg of a right triangle increases at a rate of 2 in/sec, while the other leg decreases at 3 in/sec, find how fast the hypotenuse is changing when the first leg is 6 ft and the other leg is 8 ft.

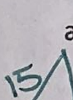
x = Leg 1
y = Leg 2

K: $dx/dt = 2$ in/sec
$dy/dt = -3$ in/sec
F: dh/dt
W: $x = 6$ ft $y = 8$ ft $h = 10$ ft
or $x = 72$ in $y = 96$ in $h = 120$ ft

$$x^2 + y^2 = h^2$$
$$2x\,dx/dt + 2y\,dy/dt = 2h\,dh/dt$$
$$\cancel{2}(72)(2) + \cancel{2}(96)(-3) = \cancel{2}(120)\,dh/dt$$
$$-144 = 120\,dh/dt$$
$$\frac{dh}{dt} = \frac{-6}{5} \text{ in/sec}$$

4. A ladder 15 m tall slides down the side of a water tower. When the bottom end is 11 m from the tower, the opposite end is sliding down at a rate of 3 m/h.

a. At that instant, how fast is the bottom of the ladder moving away from the tower?



$11^2 + y^2 = 15^2$
$y^2 = 104$
$y = 2\sqrt{26}$

K: $dy/dt = -3$ m/h
F: dx/dt
W: $x = 11$ $y = 2\sqrt{26}$

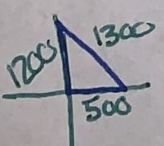
$$x^2 + y^2 = L^2$$
$$x\,dx/dt + y\,dy/dt = 0$$
$$11\,dx/dt + 2\sqrt{26}(-3) = 0$$
$$\frac{dx}{dt} = \frac{6\sqrt{26}}{11} \text{ m/hr}$$

b. How fast is the area of the region created between the ladder, the ground, and the tower changing?

K: $\frac{dy}{dt} = -3$ m/h $\frac{dx}{dt} = \frac{6\sqrt{26}}{11}$
F: dA/dt
W: $x = 11$ m $y = 2\sqrt{26}$ m

$$A = \tfrac{1}{2}xy$$
$$dA/dt = \tfrac{1}{2}x\frac{dy}{dt} + \tfrac{1}{2}y\frac{dx}{dt}$$
$$dA/dt = \tfrac{1}{2}(11)(-3) + \tfrac{1}{2}(2\sqrt{26})\left(\frac{6\sqrt{26}}{11}\right)$$
$$\frac{dA}{dt} = \frac{-33}{2} + \frac{156}{11}$$
$$\frac{dA}{dt} = \frac{-51}{22} \text{ m}^2\text{/hr}$$

5. Darth Vader's spaceship is approaching the origin along the positive y axis at 50 km/sec. Meanwhile, his daughter Ella's spaceship is moving away from the origin along the positive x-axis at 80 km/sec. When Darth is at $y = 1200\ km$ and Ella is at $x = 500\ km$, is the distance between them increasing or decreasing? At what rate?

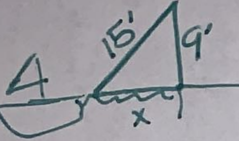


K: $\frac{dy}{dt} = -50$ Km/sec
$\frac{dx}{dt} = 80$ Km/sec
F: $\frac{dd}{dt}$
W: $x = 500$ Km
$y = 1200$ Km

$$x^2 + y^2 = d^2$$
$$x\frac{dx}{dt} + y\frac{dy}{dt} = d\frac{dd}{dt}$$
$$500(80) + 1200(-50) = 1300\left(\frac{dd}{dt}\right)$$
$$40{,}000 - 60{,}000 = 1300\,dd/dt$$
$$\frac{-20{,}000}{1300} = dd/dt$$
$$dd/dt = \frac{-200}{13} \text{ Km/sec}$$

The distance between them is decreasing

6. A winch at the end of the dock is 9 ft above the level of the deck of a boat. A rope attached to the deck is being hauled in by the winch at a rate of 3 ft/sec. How fast is the boat being pulled toward the dock when 15 ft of rope are out?



K: $dr/dt = -3$ ft/sec

F: dx/dt

W: $x = 12'$ $y = 9'$

$r = 15'$

$x^2 + y^2 = r^2$

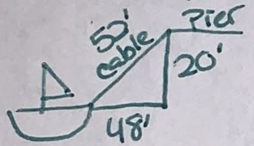
$x\,dx/dt + y\,dy/dt = r\,dr/dt$

$12(dx/dt) + 9(0) = 15(-3)$

$dx/dt = -45/12$

$dx/dt = -15/4$ ft/sec

7. A boat is pulled toward a pier by means of a taut cable. If the boat is 20 ft below the level of the pier and the cable is pulled in at a rate of 36 ft/min, how fast is the boat moving when it is 48 ft from the base of the pier?



$20^2 + 48^2 = c^2$

$c = 52$

$x^2 + y^2 = c^2$

$x\frac{dx}{dt} + y\frac{dy}{dt} = c\frac{dc}{dt}$

$48(\frac{dx}{dt}) + 20(0) = 52(-36)$

$\frac{dx}{dt} = -39$ ft/min

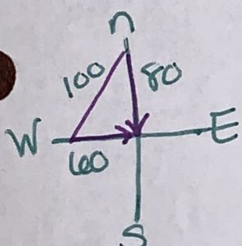
K: $\frac{dc}{dt} = -36$ ft/min

F: $\frac{dx}{dt}$

W: $x = 48$ ft, $y = 20$ft

$c = 52$ft

8. Two vehicles are approaching an intersection, one truck from the west at 15 m/sec and one van from the north at 20 m/sec. How fast is the distance between the vehicles changing at the instant the truck is 60 m west and the van 80 m north of the intersection?



K: $\frac{dx}{dt} = -15$m/sec

$\frac{dy}{dt} = -20$m/sec

F: dd/dt

W: $x = 60$m $y = 80$m

$d = 100$m

$x^2 + y^2 = d^2$

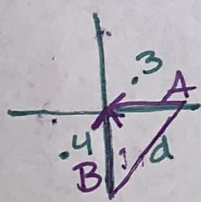
$x\,dx/dt + y\,dy/dt = d\,dd/dt$

$60(-15) + 80(-20) = 100\,dd/dt$

$-2500 = 100\,dd/dt$

$dd/dt = -25$ m/sec

9. Car A is going west at 50 mph and car B is headed north at 60 mph. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?



K: $\frac{dA}{dt} = -50$mph

$\frac{dB}{dt} = -60$mph

F: $\frac{dd}{dt}$

W: $A = .3$mi $B = .4$mi

$d = .5$mi

$A^2 + B^2 = d^2$

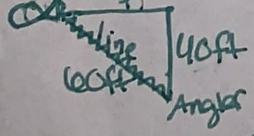
$A\,dA/dt + B\,dB/dt = d\,dd/dt$

$(.3)(-50) + (.4)(-60) = .5\,dd/dt$

$-39 = .5\,dd/dt$

$dd/dt = -78$mph

10. An angler has hooked a fish. The fish was swimming in an east-west direction along a line 40 ft north of the angler. If the line is leaving the reel at a rate of 7 ft/sec when the fish is 60 ft from the angler, how fast is the fish traveling?



$f^2 + 40^2 = 60^2$

$f^2 = \sqrt{2000}$

$f = 20\sqrt{5}$

K: $\frac{dL}{dt} = 7$ft/sec

F: $\frac{df}{dt}$

W: $y = 40'$

$L = 60'$

$f = 20\sqrt{5}'$

$f^2 + y^2 = L^2$

$f\frac{df}{dt} + y\frac{dy}{dt} = L\frac{dL}{dt}$

$20\sqrt{5}\,df/dt + 40(0) = 60(7)$

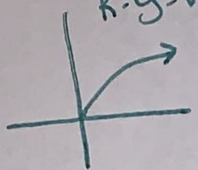
$\frac{df}{dt} = \frac{420}{20\sqrt{5}}$

$\frac{df}{dt} = \frac{21}{\sqrt{5}}$ ft/sec

Related Rates – Moving Particles, Angles, Formulas

1. A particle is moving on the graph of $y = \sqrt{x}$. At what point on the curve are the x-coordinate and y-coordinate of the particle changing at the same rate?

K: $y=\sqrt{x}$



F: (x,y)

W: $\frac{dx}{dt} = \frac{dy}{dt}$

$y = \sqrt{x}$

$\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt}$

$1 = \frac{1}{2\sqrt{x}}$

$2\sqrt{x} = 1$

$\sqrt{x} = \frac{1}{2}$

$x = \frac{1}{4}$

$y = \sqrt{x}$

$y = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$(1/4, 1/2)$

2. Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$, if $\frac{dx}{dt} = 2\ cm/sec$.

K: $\frac{dx}{dt} = 2cm/sec$

F: dd/dt

$(\underset{x_1}{0},\underset{y_1}{0}) \rightarrow (\underset{x_2}{x}, \underset{y_2}{x^2+1})$

W: $y = x^2+1$

$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$d = \sqrt{(x-0)^2+(x^2+1-0)^2}$

$d = \sqrt{x^2+x^4+2x^2+1}$

$d = \sqrt{x^4+3x^2+1}$

$dd/dt = \frac{1}{2\sqrt{x^4+3x^2+1}}\left(4x^3\frac{dx}{dt}+6x\frac{dx}{dt}\right)$

$dd/dt = \frac{4x^3+6x}{\sqrt{x^4+3x^2+1}}$

3. A particle moves along the curve $y = \sqrt{1+x^3}$. As it reaches the point (2,3), the y-coordinate is increasing at a rate of 4 cm/sec. How fast is the x-coordinate of the point changing at that instant?

K: $\frac{dy}{dt} = 4cm/sec$

F: $\frac{dx}{dt}$

W: $(\underset{x}{2},\underset{y}{3})$

$y = \sqrt{1+x^3}$

$\frac{dy}{dt} = \frac{1}{2}(1+x^3)^{-1/2}\cdot 3x^2\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{3x^2}{2\sqrt{1+x^3}}\frac{dx}{dt}$

$4 = \frac{3(2)^2}{2\sqrt{1+2^3}}\frac{dx}{dt}$

$4 = \frac{12}{2\sqrt{9}}\frac{dx}{dt}$

$4 = 2\frac{dx}{dx}$

$\frac{dx}{dt} = 2cm/sec$

4. A particle moves along a path described by $y = 4 - x^2$. At what point along the curve are the x and y values changing at the same rate?

K: $\frac{dx}{dt} = \frac{dy}{dt}$

F: (x,y)

W: $y = 4-x^2$

$y = 4-x^2$

$\frac{dy}{dt} = -2x\frac{dx}{dt}$

$1 = -2x$

$x = -\frac{1}{2}$

$y = 4-(-\frac{1}{2})^2 = \frac{15}{4}$

$(-\frac{1}{2}, \frac{15}{4})$

5. A particle is moving along the curve $y = x \cdot \ln x$. Find all values of x at which the rate of change of y with respect to time is 3 times that of x. (Assume $\frac{dx}{dt}$ is never 0).

K: $\frac{dy}{dt} = 3\cdot\frac{dx}{dt}$

F: x

W: $y = x\cdot\ln x$

$y = x\ln x$

$\frac{dy}{dt} = x\cdot\frac{1}{x}\frac{dx}{dt} + \ln x\frac{dx}{dt}$

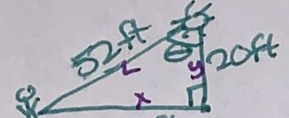
$3\frac{dx}{dt} = \frac{dx}{dt} + \ln x\frac{dx}{dt}$

$3 = 1 + \ln x$

$2 = \ln x$

$x = e^2$

6. A man walks along a straight path at a rate of 4 ft/sec. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 48 feet from the point on the path closest to the searchlight?



$20^2+48^2=L^2$

$L=52'$

K: $\frac{dx}{dt}=4$ ft/sec

F: $\frac{d\theta}{dt}$

W: $x=48'$, $y=20'$, $L=52'$

$\sec\theta=\frac{52}{20}=\frac{13}{5}$

$\tan\theta=\frac{x}{20}$ ← constant

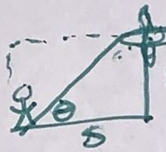
$20\tan\theta=x$

$20\sec^2\theta\frac{d\theta}{dt}=\frac{dx}{dt}$

$20\left(\frac{13}{5}\right)^2\frac{d\theta}{dt}=4$

$\frac{20\cdot 169}{25}\cdot\frac{d\theta}{dt}=\frac{4\cdot 5}{4\cdot 169}=\frac{5}{169}$ rad/sec

7. An airplane is flying at a constant speed at an altitude of 10,000 ft on a line that will take it directly over an observer on the ground. At a given instant, the observer notes that the angle of elevation is $\pi/3$ rad. and is increasing at the rate of 1/60 rad/sec. Find the speed of the plane. (Find the rate at which the plane's position is changing).



K: $\frac{d\theta}{dt}=\frac{1}{60}$ rad/sec

F: $\frac{ds}{dt}$

W: $\theta=\frac{\pi}{3}$

$\csc\frac{\pi}{3}=\frac{2}{\sqrt{3}}$

$\cot\theta=\frac{s}{10{,}000}$

$10{,}000\cot\theta=s$

$-10{,}000\csc^2\theta\frac{d\theta}{dt}=\frac{ds}{dt}$

$-10{,}000\left(\frac{2}{\sqrt{3}}\right)^2\frac{1}{60}=\frac{ds}{dt}$

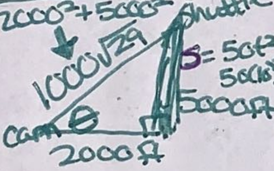
$\frac{ds}{dt}=-\frac{2000}{9}$ ft/sec

8. Find the rate of change in the angle of elevation of the camera filming the lift off of the space shuttle 10 seconds after lift-off. The camera is located 2000 ft from the base of the shuttle and the shuttle is rising vertically according to the position equation $s=50t^2$ s us measured in feet and t is measured in seconds.

$\sqrt{2000^2+5000^2}$

↓

$1000\sqrt{29}$



K: $\frac{ds}{dt}=100t\,\frac{dt}{dt}=100(10)=1000$ ft/sec

F: $d\theta/dt$

W: $t=10$ sec

$s=5000$

$\tan\theta=\frac{s}{2000}$

$\sec^2\theta\frac{d\theta}{dt}=\frac{1}{2000}\frac{ds}{dt}$

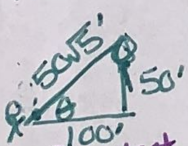
$\left(\frac{\sqrt{29}}{2}\right)^2\frac{d\theta}{dt}=\frac{1000}{2000}\rightarrow\frac{d\theta}{dt}=\frac{1}{2}\cdot\frac{4}{29}=\frac{2}{29}$ rad/sec

$\sec\theta=\frac{1000\sqrt{29}}{2000}$

$\sec\theta=\frac{\sqrt{29}}{2}$

9. A balloon rises vertically at a rate of 10 ft/sec. Joe watches the balloon ascend from a point on the ground 100 ft away from the spot below the rising balloon. At what rate is Joe's eye rotating upward to follow the balloon when the balloon is 50 ft above the level of Joe's eye?

$\sqrt{50^2+100^2}$



K: $\frac{dy}{dt}=10$ ft/sec

F: $\frac{d\theta}{dt}$

W: $y=50$ ft

$\sec\theta=\frac{50\sqrt{5}}{100}=\frac{\sqrt{5}}{2}$

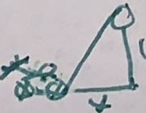
$\tan\theta=\frac{y}{100}$

$\sec^2\theta\frac{d\theta}{dt}=\frac{1}{100}\frac{dy}{dt}$

$\left(\frac{\sqrt{5}}{2}\right)^2\frac{d\theta}{dt}=\frac{10}{100}$

$\frac{5}{4}\quad\frac{d\theta}{dt}=\frac{1}{10}\cdot\frac{4}{5}=\frac{2}{25}$ rad/sec

10. A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicyclist moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and the balloon increasing 3 seconds later?



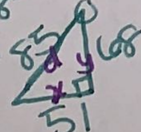
K: $\frac{dy}{dt}=1$ ft/sec

$\frac{dx}{dt}=17$ ft/sec

F: $\frac{dd}{dt}$

$\sqrt{51^2+68^2}$

→ 85



W: $t=3$ sec

so $y=65+1\text{ft}(3\text{sec})$

$y=68$ ft

$x=17(3)=51$ ft

$x^2+y^2=d^2$

$x\frac{dx}{dt}+y\frac{dy}{dt}=d\frac{dd}{dt}$

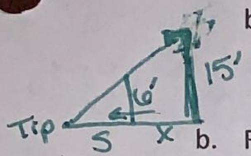
$51(17)+68(1)=85\frac{dd}{dt}$

$935=85\frac{dd}{dt}$

$\frac{dd}{dt}=11$ ft/sec

Related Rates Using Similar Triangles – Shadows, Cones, Coffee Pots, Troughs

1. A streetlight is 15 feet above the sidewalk. A man 6 feet tall walks away from the light at the rate of 5 ft/sec.
   a. Determine the rate at which the man's shadow is lengthening at the moment that he is 20 feet from the base of the light.



K: $dx/dt = 5$ ft/sec
F: ds/dt
W: $x = 20$ ft

$\frac{6}{s} = \frac{15}{x+s}$

$15s = 6x + 6s$

$9s = 6x$

$9\frac{ds}{dt} = 6\frac{dx}{dt}$

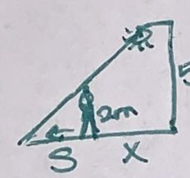
$\frac{ds}{dt} = \frac{6(5)}{9} = \frac{10}{3}$ ft/sec

   b. Find the rate at which the tip of the shadow is changing at this time.

$\frac{dtip}{dt} = \frac{dx}{dt} + \frac{ds}{dt}$

$\frac{dtip}{dt} = 5 + \frac{10}{3} = \frac{25}{3}$ ft/sec

2. A man 2 m tall walks away from a lamppost whose light is 5 m above the ground. If he walks at a speed of 1.5 m/s, at what rate is his shadow growing when he is 10 m from the lamppost?



K: $dx/dt = 1.5$ m/s
F: ds/dt
W: $x = 10$m

$\frac{2}{s} = \frac{5}{x+s}$

$5s = 2x + 2s$

$3s = 2x$

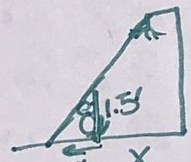
$s = \frac{2}{3}x$

$\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt}$

$\frac{ds}{dt} = \frac{2}{3} \cdot \frac{3}{2}$

$\frac{ds}{dt} = 1$ m/s

3. Sulley the squirrel, a stunning 1.5 ft tall, is walking away from a 15 ft lamppost at a rate of 6 ft/min and heading home after collecting nuts for the winter. How fast is the length of Sulley's shadow increasing? At what rate is the tip of his shadow changing?



K: $dx/dt = 6$ ft/min
F: ds/dt + $\frac{dtip}{dt}$
W:

$\frac{1.5}{s} = \frac{15}{x+s}$

$15s = 1.5x + 1.5s$

$15s = \frac{3}{2}x + \frac{3}{2}s$

$30s = 3x + 3s$

$27s = 3x$

$s = \frac{1}{9}x$

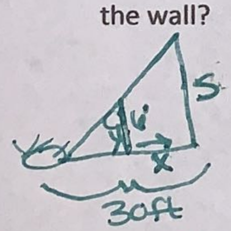
$\frac{ds}{dt} = \frac{1}{9}\frac{dx}{dt}$

$\frac{ds}{dt} = \frac{1}{9} \cdot 6 = \frac{2}{3}$ ft/min

$\frac{dtip}{dt} = \frac{dx}{dt} + \frac{ds}{dt}$

$\frac{dtip}{dt} = 6 + \frac{2}{3} = \frac{20}{3}$ ft/min

4. A man 6 ft tall walks toward a wall. A light, 30 ft from the wall, is on the ground directly behind the man. If the man is walking at a rate of 4 ft/sec, how fast is the tip of the shadow moving up the wall when he is 5 feet from the wall?



K: $\frac{dx}{dt} = -4$ ft/sec
F: $\frac{ds}{dt}$
W: $x = 5$ft

$\frac{s}{30} = \frac{6}{25} \to s = \frac{180}{25} = \frac{36}{5}$ ft when $x = 5'$

$\frac{s}{30} = \frac{6}{30-x}$

$180 = 30s - xs$

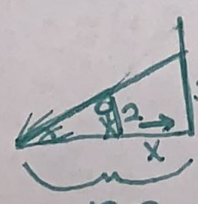
$0 = 30\frac{ds}{dt} - x\frac{ds}{dt} - s\frac{dx}{dt}$

$0 = 30\frac{ds}{dt} - 5\frac{ds}{dt} - \left(\frac{36}{5}\right)(-4)$

$-25\frac{ds}{dt} = \frac{144}{5}$

$\frac{ds}{dt} = \frac{-144}{125}$ ft/sec.

5. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight towards the building at a speed of 1.6 m/sec, how fast is his shadow on the building decreasing when he is 4 meters from the building?



K: $\frac{dx}{dt} = -1.6$ m/s
F: $\frac{ds}{dt}$
W: $x = 4$m

$s = 3$m

$\frac{s}{12} = \frac{2}{8}$

$8s = 24$

$\frac{s}{12} = \frac{2}{12-x}$

$24 = 12s - xs$

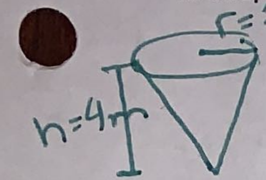
$0 = 12\frac{ds}{dt} - x\frac{ds}{dt} - s\frac{dx}{dt}$

$0 = 12\frac{ds}{dt} - 4\frac{ds}{dt} - 3(-1.6)$

$-8\frac{ds}{dt} = 4.8$

$\frac{ds}{dt} = -\frac{3}{5}$ m/sec

6. A water tank has the shape of an inverted circular cone with base radius 2 m and a height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



K: $\frac{dV}{dt} = 2 m^3/min$

F: $\frac{dh}{dt}$

W: $h = 3$

$\frac{h}{r} = \frac{4}{2}$

$4r = 2h$

$r = \frac{1}{2}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi (\frac{1}{2}h)^2 h$

$V = \frac{\pi}{12}h^3$

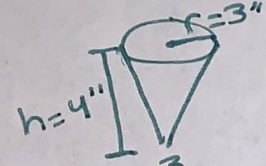
$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$

$2 = \frac{\pi}{4}(3)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = 2 \cdot \frac{4}{9\pi}$

$\frac{dh}{dt} = \frac{8}{9\pi}$ m/min

7. Water is flowing into an inverted cone at the rate of 5 cubic inches per second. If the cone has an altitude of 4 in and a base radius of 3 in, how fast is the water level rising when the water is 2 in deep? How fast is the radius of the water changing hen the water is 2 in deep?



K: $\frac{dV}{dt} = 5 in^3/sec$

F: $\frac{dh}{dt}$, $\frac{dr}{dt}$

W: $h = 2"$

$\frac{r}{h} = \frac{3}{4}$

$3h = 4r$

$h = \frac{4}{3}r \rightarrow r = \frac{3}{4}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi (\frac{3}{4}h)^2 h$

$V = \frac{3\pi}{16}h^3$

$\frac{dV}{dt} = \frac{9\pi}{16}h^2 \frac{dh}{dt}$

$5 = \frac{9\pi}{16}(4)(\frac{dh}{dt})$

$\frac{dh}{dt} = \frac{20}{9\pi}$ in/sec

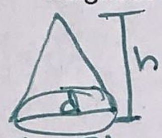
$r = \frac{3}{4}h$

$\frac{dr}{dt} = \frac{3}{4}\frac{dh}{dt}$

$\frac{dr}{dt} = \frac{3}{4} \cdot \frac{20}{9\pi}$

$\frac{dr}{dt} = \frac{5}{3\pi}$ in/sec

8. At a sand and gravel plant, sand is falling off a conveyer and into a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high?



$d = 3h$

K: $\frac{dV}{dt} = 10 ft^3/min$

F: $\frac{dh}{dt}$

W: $h = 15$

$V = \frac{1}{3}\pi (\frac{d}{2})^2 h$

$V = \frac{1}{3}\pi (\frac{3}{2}h)^2 h$

$V = \frac{3\pi}{4}h^3$

$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \frac{dh}{dt}$

$10 = \frac{9\pi}{4}(225)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{40}{2025\pi} = \frac{8}{405\pi}$ ft/min

9. Coffee is draining from a conical filter into a cylindrical coffeepot at a rate of 10 in^3/min. K: $\frac{dV}{dt} = +10 in^3/min$ F: $\frac{dh}{dt}$ W: $h = 5"$

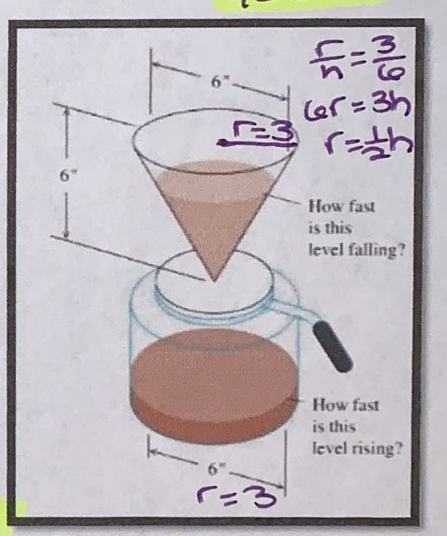
a. How fast is the level of the coffee in the pot rising when the coffee in the cone is 5 in deep? ★r in coffee pot is constant $r = 3"$

$V = \pi r^2 h$

$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

$10 = \pi (3)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{10}{9\pi}$ in/min



$\frac{r}{h} = \frac{3}{6}$

$6r = 3h$

$r = \frac{1}{2}h$

$r = 3$

$r = 3$

K: $\frac{dV}{dt} = -10$ in³/min

F: $\frac{dh}{dt}$

W: $h = 5"$

b. How fast is the level in the cone falling at that moment?

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi (\frac{1}{2}h)^2 h$

$V = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$

$-10 = \frac{\pi}{4}(5)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{-40}{25\pi} = \frac{-8}{5\pi}$ in/min

10. A trough is 15 ft long and 4 ft across the top as shown in the figure to the right. Its ends are isosceles triangles with a height of 3 ft. Water runs into the trough at the rate of 2.5 ft^3/min.

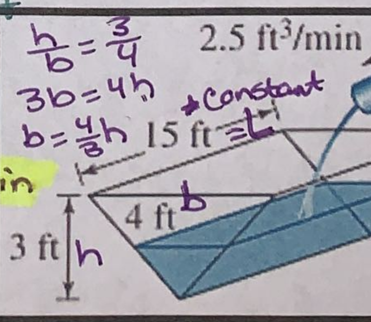
K: $\frac{dV}{dt} = 2.5 ft^3/min$ F: $\frac{dh}{dt}$ W: $h = 2 ft$ $L = 15 ft$

$b = \frac{4}{3}h$

$\frac{h}{b} = \frac{3}{4}$

$3b = 4h$

$b = \frac{4}{3}h$



a. How fast is the water level rising when it is 2 ft deep?

$V = \frac{1}{2}bhL$

$V = \frac{1}{2}(\frac{4}{3}h)h(15)$

$V = 10h^2$

$\frac{dV}{dt} = 20h \frac{dh}{dt}$

$2.5 = 20(2)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{16}$ ft/min

b. How fast is the surface area changing when the water level is 2 ft deep?

$A = bL$

$A = \frac{4}{3}h \cdot 15$

$A = 20h$

$\frac{dA}{dt} = 20\frac{dh}{dt}$

$\frac{dA}{dt} = 20(\frac{1}{16})$

$\frac{dA}{dt} = \frac{5}{4}$ ft²/min

Extreme Value Theorem and Max/Min Values

Determine the absolute maximum and absolute minimum value over the stated interval by applying the Extreme Value Theorem.

1. $f(x) = x^3 - 12x$ $(0,4)$

f(x) is cont.

$f'(x) = 3x^2 - 12$

$0 = x^2 - 4$

$x = \pm 2$

$f(0) = 0$

$f(2) = -16$ abs min.

$f(4) = 16$ abs. max

2. $f(x) = \frac{x}{x-2}$ $[3,5]$ f(x) is cont. on [3,5]

$f'(x) = \frac{x-2-x}{(x-2)^2}$

$0 = \frac{-2}{(x-2)^2}$

No sol.: no critical values

$f(3) = 3$ Abs. max

$f(5) = 5/3$ Abs. min.

3. $f(x) = \frac{1}{x}$ $[-1,3]$

f(x) is not cont. over [-1,3]

x = 0 is a V.A.

4. $f(x) = \frac{1}{1+x^2}$ $(-3,3)$ $f(x) = (1+x^2)^{-1}$

f(x) is continuous

$f'(x) = \frac{-1 \cdot 2x}{(1+x^2)^2}$

$0 = \frac{-2x}{(1+x^2)^2}$

$x = 0$

$f(-3) = 1/10$ Abs. min.

$f(0) = 1$ Abs. max.

$f(3) = 1/10$ Abs. min.

5. $f(x) = \sqrt[3]{x}$ $[-1,27]$

f(x) is continuous

$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$

$0 = \frac{1}{3\sqrt[3]{x^2}}$

no sol :. no critical values

$f(-1) = -1$ Abs min.

$f(27) = 3$ Abs. max.

6. $f(x) = \sqrt{9 - x^2}$ $[-1,2]$

f(x) is cont. on [-1,2]

$f'(x) = \frac{-2x}{2\sqrt{9-x^2}}$

$0 = \frac{-2x}{2\sqrt{9-x^2}}$

$x = 0$

$f(-1) = \sqrt{8} = 2\sqrt{2}$

$f(0) = 3$ Abs. max.

$f(2) = \sqrt{5}$ Abs. min.

Mean Value Theorem

For the following functions, determine whether the Mean Value Theorem applies on the given closed interval. If the Mean Value Theorem applies, state why it applies and find the value(s) of c that satisfies the MVT. If the Mean Value Theorem does not apply, state why.

1. $f(x) = 5 - \frac{4}{x}$ on $[1,4]$

f(x) is cont. on [1,4]

f(x) is diff. on (1,4)

$m_{sec} = \frac{f(4)-f(1)}{4-1} = \frac{4-1}{4-1} = 1$

$f'(x) = \frac{4}{x^2}$

$1 = \frac{4}{x^2}$

$x^2 = 4$

$x = \pm 2$

$\therefore c = 2$

2. $f(x) = \sqrt{2 - x}$ on $[-7,2]$

f(x) is cont. on [-7,2]

f'(x) is diff. on (-7,2)

$m_{sec} = \frac{f(2)-f(-7)}{2--7} = \frac{0-3}{2+7} = -\frac{1}{3}$

$f'(x) = \frac{-1}{2\sqrt{2-x}}$

$-\frac{1}{3} = \frac{-1}{2\sqrt{2-x}}$

$-3 = -2\sqrt{2-x}$

$9 = 4(2-x)$

$9 = 8 - 4x$

$1 = -4x$

$x = -1/4$

$\therefore c = -\frac{1}{4}$

3. $f(x) = \frac{x^2-1}{x-2}$ on $[-1,3]$

f(x) is not cont. on [-1,3]

bc there is a VA @ x = 2

4. $f(x) = \frac{x^2-1}{x}$ on $[-1,1]$

f(x) is not continuous on [-1,1]

bc there is a V.A. @ x = 0

5. $f(x) = \sin x$ on $[0, \pi]$

f(x) is continuous on [0, π]

f'(x) is differentiable on (0, π)

$m_{sec} = \frac{f(\pi)-f(0)}{\pi - 0} = \frac{0-0}{\pi} = 0$

$f'(x) = \cos x$

$\cos x = 0$

$x = \pi/2$

$\therefore c = \pi/2$

6. $f(x) = \frac{x+1}{x}$ on $\left[\frac{1}{2}, 2\right]$

f(x) is cont. on [1/2, 2]

f(x) is diff. on (1/2, 2)

$m_{sec} = \frac{f(2)-f(1/2)}{2-1/2} = \frac{3/2 - 3}{3/2} = -\frac{3}{2} \cdot \frac{2}{3} = -1$

$f'(x) = \frac{x - x - 1}{x^2}$

$f'(x) = \frac{-1}{x^2}$

$-\frac{1}{x^2} = -1$

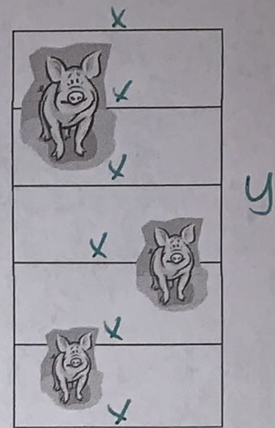
$x^2 = 1$

$x = \pm 1$

$\therefore c = 1$

Optimization

1. A pig farmer has 600 meters of fencing with which to enclose and divide 5 adjacent rectangular pens, as shown in the figure. What dimensions will result in the maximum possible total area for the five pens.



$600 = 6x + 2y$

$2y = 600 - 6x$

$y = 300 - 3x$

$A = xy$

$A = x(300 - 3x)$

$A = 300x - 3x^2$

$A' = 300 - 6x = 0$

$6x = 300$

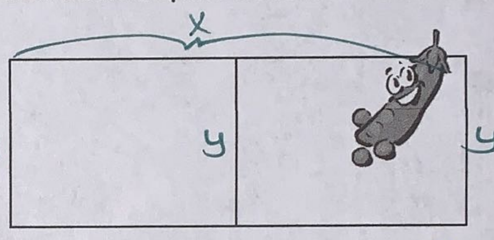
$x = 50m$

$y = 300 - 3(50)$

$y = 150m$

50m x 150m

2. A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two parts by another fence parallel to one of the sides. What are the dimensions of the outer rectangle that would require the least amount of fence?



$A = xy$

$216 = xy$

$y = \frac{216}{x}$

$P = 2x + 3y$

$P = 2x + 3\left(\frac{216}{x}\right)$

$P = 2x + \frac{648}{x}$

$P' = 2 - \frac{648}{x^2} = 0$

$\frac{648}{x^2} = 2$

$2x^2 = 648$

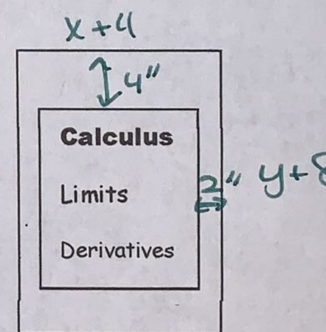
$x^2 = 324$

$x = 18$

$y = \frac{216}{18} = 12$

18m x 12m

3. You are designing a poster to contain 50 in^2 of printing with a 4-in margin at the top and bottom and a 2-in margin on each side. What overall dimensions will minimize the size of the poster?



$A = xy$

$50 = xy$

$y = \frac{50}{x}$

$A = xy$

$A = (x+4)(y+8)$

$A = (x+4)\left(\frac{50}{x}+8\right)$

$A = 50 + 8x + \frac{200}{x} + 32$

$A = 8 - \frac{200}{x^2}$

$\frac{200}{x^2} = 8$

$8x^2 = 200$

$x^2 = 25$

$x = 5"$

$y = 10"$

$x + 4 = 5 + 4 = 9"$

$y + 8 = 10 + 8 = 18"$

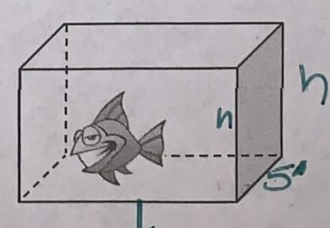
min

$-$ $+$

5

9" x 18"

4. You are building a glass fish tank that will hold 75ft³ of water. You want its base and sides to be rectangular and the top to be open. So that the tank will fit on the shelf in your room the width must be 5 feet, but the length and height can vary. Building materials for the tank cost \$10 per square foot for the base and \$3 per square foot for the sides. What are the dimensions of the tank with the minimum cost? What is the cost of the least expensive tank?



$C = 10(5L) + 3(2 \cdot 5h) + 3(2 \cdot hL)$

$C = 50L + 30h + 6hL$

$C = 50L + 30\left(\frac{15}{L}\right) + 6\left(\frac{15}{L}\right)L$

$C = 50L + \frac{450}{L} + 90$

$C' = 50 - \frac{450}{L^2}$

$\frac{450}{L^2} = 50$

$50L^2 = 450$

$L^2 = 9$ so $L = 3ft$

$V = LWH$

$75 = 5LH$

$15 = LH$

$h = \frac{15}{L}$

$h = \frac{15}{3}$

$h = 5ft$

3ft x 5ft x 5ft

$C = 10(5 \cdot 3) + 3(2 \cdot 5 \cdot 5) + 3(2 \cdot 5 \cdot 3)$

$C = \$390$

5. A rectangle is inscribed in the region bounded by one arch of a cosine curve and the x-axis. What value of x gives the maximum area? What is the maximum area? (calculator needed)

$y = \cos x$

$A = 2xy$

$A = 2x \cdot \cos x$

$A' = -2x\sin x + 2\cos x$

$0 = -2x\sin x + 2\cos x$

$2\cos x = 2x\sin x$

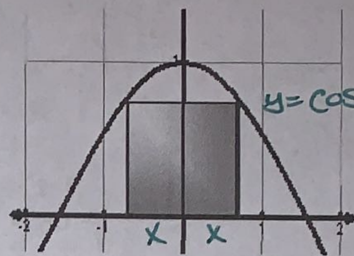
$x = 0.8603$

$y = \cos(.8603)$

$y = 0.65221$

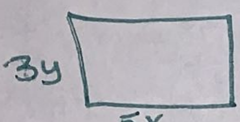
$A = 2(.8603)(0.6522)$

$A = 1.122u^2$



.8603

6. A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy-duty fencing selling for $5 a foot, while the remaining two sides will use standard fencing selling for $3 a foot. What are the dimensions of the rectangular plot of greatest area that can be fenced in at a cost of $6000?



$6000 = 2(5x) + 2(3y)$

$6000 = 10x + 6y$

$y = \frac{6000 - 10x}{6}$

$y = 1000 - \frac{5}{3}x$

$A = xy$

$A = x(1000 - \frac{5}{3}x)$

$A = 1000x - \frac{5}{3}x^2$

$A' = 1000 - \frac{10}{3}x$

$\frac{10}{3}x = 1000$

$10x = 3000$

$x = 300'$

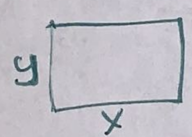
$y = 1000 - \frac{5}{3}(300)$

$y = 1000 - 500$

$y = 500'$

300ft x 500ft

7. A rectangular area of 3200 ft² is to be fenced off. Two opposite sides will use fencing costing $1 per foot and the remaining sides will use fencing costing $2 per foot. Find the dimensions of the rectangle that will cost the least.



x = $1/ft

y = $2/ft

$A = xy$

$3200 = xy$

$y = \frac{3200}{x}$

$y = \frac{3200}{80}$

$y = 40$

$C = 1x + 2y$

$C = x + 2\left(\frac{3200}{x}\right)$

$C = x + \frac{6400}{x}$

$C' = 1 - \frac{6400}{x^2} = 0$

$x^2 = 6400$

$x = 80$

80ft x 40ft

8. What is the largest area a rectangle can have inscribed in a closed region bounded by the x-axis, y-axis and the line $y = -4x + 8$?

$A = x \cdot y$

$A = x(-4x+8)$

$A = -4x^2 + 8x$

$A' = -8x + 8$

$0 = -8x + 8$

$x = 1$

max

+ | −

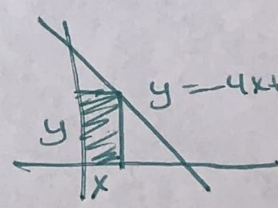
1

$y = -4(1) + 8$

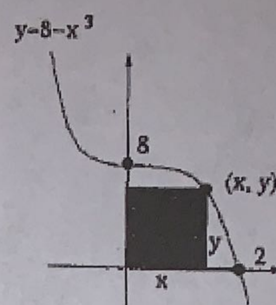
$y = 4$

$A = 1(4)$

$A = 4u^2$



9. Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y = 8 - x^3$.



$A = xy$

$A = x(8-x^3)$

$A = 8x - x^4$

$A' = 8 - 4x^3$

$0 = 4(x^3-2)$

$x = \sqrt[3]{2}$

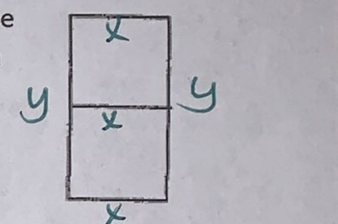
$y = 8 - (\sqrt[3]{2})^3$

$y = 8 - 2$

$y = 6$

$\sqrt[3]{2}$ u x 6u

10. A rectangle field is to have area 60,000 m^2. Fencing is required to enclose the field and to divide it in half (two equal areas). What are the outer dimensions of the field that require the minimum amount of fencing?



$60{,}000 = xy$

$y = \frac{60{,}000}{x}$

$P = 3x + 2y$

$P = 3x + 2\left(\frac{60{,}000}{x}\right)$

$P = 3x + \frac{120{,}000}{x}$

$P' = 3 - \frac{120{,}000}{x^2}$

$\frac{120{,}000}{x^2} = 3$

$3x^2 = 120{,}000$

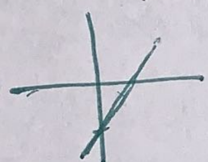
$x^2 = 40{,}000$

$x = 200$

$y = \frac{60{,}000}{200} = 300$

200m x 300m

11. Find the point on the line $y = 2x - 3$ that is the closest to the origin.



$(x, 2x-3)$

$d = \sqrt{x^2 + (2x-3)^2}$

$d = \sqrt{x^2 + 4x^2 - 12x + 9}$

$d = \sqrt{5x^2 - 12x + 9}$

$d' = \frac{10x - 12}{2\sqrt{5x^2 - 12x + 9}}$

$0 = 10x - 12$

$x = 6/5$

min

$6/5$

$y\left(\frac{6}{5}\right) = 2\left(\frac{6}{5}\right) - 3$

$= \frac{12}{5} - 3$

$= -3/5$

$(6/5, -3/5)$

12. Find the point on the parabola $y^2 = 2x$ that is closest to the point (1,4)

$x = \frac{y^2}{2}$

$(1, 4)$

$x_1\ y_1$

$\left(\frac{y^2}{2}, y\right)$

$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}$

$d' = 2\left(\frac{y^2}{2} - 1\right) \cdot y + 2(y-4)$

$d' = y^3 - 2y + 2y - 8$

$d' = y^3 - 8$

$0 = y^3 - 8$

$y = 2$

$x = \frac{y^2}{2}$

$x = \frac{2^2}{2} = 2$

$(2, 2)$

13. Find all points on the curve $x = 2y^2$ closest to (0,9).

$d = \sqrt{(2y^2 - 0)^2 + (y-9)^2}$

$d = \sqrt{4y^4 + y^2 - 18y + 81}$

$d' = 16y^3 + 2y - 18$

$0 = 8y^3 + y - 9$

$1 \,|\, 8\ \ 0\ \ 1\ \ -9$

$\ \ \ \ 8\ \ 8\ \ \ 9$

$ 8\ \ 8\ \ 9\ \ 0$

$y = 1$

$y = \frac{-8 \pm \sqrt{(8)^2 - 4(8)(9)}}{16}$ ← imaginary

$-\ \ +$

1

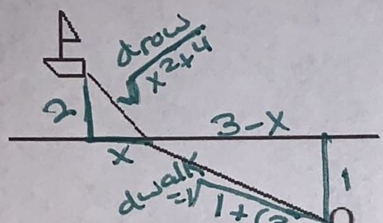
$x = 2(1)^2$

$x = 2$

$(2, 1)$

Optimization Problems – Time Problems

1. A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, three miles down the coast and 1 mile inland. If he can row at 2mph and walk at 4 mph, toward what point on the coast should he row in order to reach point Q in the least time? What if he could row at 4mph and all else stay the same?



$d = rt$

$t = \frac{d}{r}$

$T_{total} = T_{row} + T_{walk}$

$T = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{1+(3-x)^2}}{4}$

Solve $\left(\frac{d}{dx}(T(x)) = 0, x\right)$ $x = 1mi$

He should row towards 1mi down the coast if he rows at 2mph

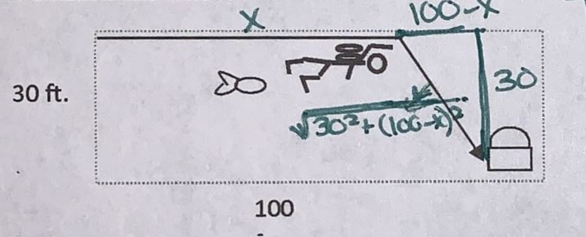
Row at 4mph:

$T = \frac{\sqrt{x^2+4}}{4} + \frac{\sqrt{1+(3-x)^2}}{4}$

$x = 2mi$

He should row towards 2mi down the coast if he can row at 4mph.

2. A scuba diver heads for a point on the bottom that is 30 m below the surface and 100 m horizontally from the point where she entered the water. She can move 13 m/min on the surface but only 12 m/min as she is descending. How far from her entry point should she start descending to reach her destination in minimum time?



$T_{total} = T_{surface} + T_{descending}$

$T = \frac{x}{13} + \frac{\sqrt{(30)^2+(100-x)^2}}{12}$

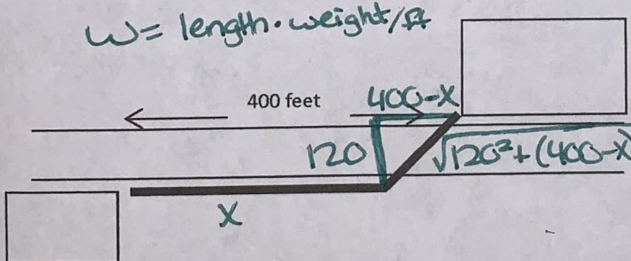
Solve $\left(\frac{d}{dx}(T(x)) = 0, x\right)$

$x = 28m$

Start diving 28m from entry point

3. A walkway is to be built from the corner of one building to the corner of another building across the street and 400 feet down the block. It is 120 feet across the street. Engineering studies show that the walkway will weigh 3000 lb/ft where it parallels the street and 4000 ft/lb where it crosses the street. How should the walkway be laid out in order to minimize its total weight?

W = length · weight/ft



$W_{total} = W_{parallel} + W_{crosses}$

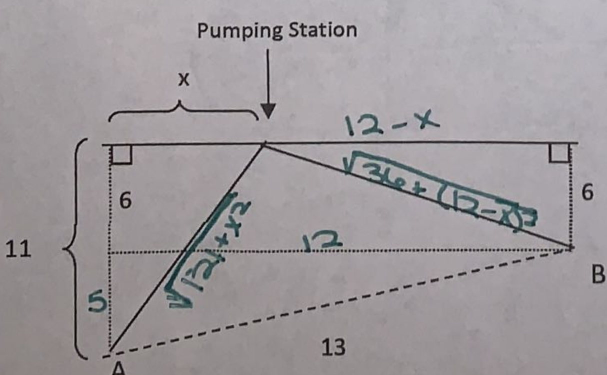
$W = 3000x + 4000\left(\sqrt{120^2+(400-x)^2}\right)$

Solve $\left(\frac{d}{dx}(W(x)) = 0, x\right)$

$x \approx 263.933$ ft parallel to street

& then 181.423 ft crossing the street

4. Town A is 11 miles from a straight river and town B is 6 miles from that same river. The distance from town A to town B is 13 miles. A pumping station is to be built along the river to supply water to both towns. Where should the pumping station be built so that the sum of the distances from the pumping station to the two towns is a minimum?



$D_{Total} = D_{Town A} + D_{Town B}$

$D = \sqrt{121+x^2} + \sqrt{36+(12-x)^2}$

Solve $\left(\frac{d}{dx}(D(x)) = 0, x\right)$

$x = \frac{132}{17} \approx 7.765$ mi down the river from Town A