4.1 The Power Rule

Power Rule: $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

Steps:
1. Rewrite the expression in the form x^{exponent}
2. Perform the Power Rule
3. Simplify ***No Negative Exponents!

Find the derivative & simplify:

1. $y = \sqrt[3]{x} + \sqrt[5]{x}$
$y = x^{1/3} + x^{1/5}$ Rewrite 1st
$y' = \frac{1}{3}x^{-2/3} + \frac{1}{5}x^{-4/5}$ PowerRule
$y' = \frac{1}{3x^{2/3}} + \frac{1}{5x^{4/5}}$

2. $y = \frac{1}{x^2}$
$y = x^{-2}$ Rewrite
$y' = -2x^{-3}$ Power Rule
$y' = \frac{-2}{x^3}$

3. $y = \frac{5}{2x^3}$
$y = \frac{5}{2}x^{-3}$
$y' = \frac{5}{2} \cdot -3x^{-4}$
$y' = \frac{-15}{2x^4}$

4. $y = \frac{1}{2\sqrt[3]{x}}$
$y = \frac{1}{2}x^{-1/3}$
$y' = \frac{1}{2} \cdot -\frac{1}{3}x^{-4/3}$
$y' = \frac{-1}{6x^{4/3}}$

5. $y = \frac{7}{3x^{-2}}$
$y = \frac{7}{3}x^2$
$y' = \frac{7}{3} \cdot 2x$
$y' = \frac{14}{3}x$

6. $y = \frac{\sqrt{x}}{x}$
$y = \frac{x^{1/2}}{x} = x^{-1/2}$
$y' = -\frac{1}{2}x^{-3/2}$
$y' = \frac{-1}{2x^{3/2}}$

7. $y = 3x\left(x^2 - \frac{2}{x}\right)$
$y = 3x^3 - 6$
$y' = 9x^2$

8. $y = \frac{x^3 - 3x^2 + 4}{x^2}$
$y = x - 3 + 4x^{-2}$
$y' = 1 - 0 - 8x^{-3}$
$y' = 1 - \frac{8}{x^3}$

9. $T(c) = 4c^2 - 7c + 3$ Find $\frac{dT}{dc}\Big|_{c=2}$
$\frac{dT}{dc} = 4 \cdot 2c - 7$
$\frac{dT}{dc} = 8c - 7$
$\frac{dT}{dc}\Big|_{c=2} = 8(2) - 7$ $\quad \frac{dT}{dc}\Big|_{c=2} = 9$

4.2 Product + Quotient Rules

Product Rule: $\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$

In words: 1st times deriv. of 2nd + 2nd times deriv of 1st

Find the derivative:

1. $h(x) = 3\sqrt{x}\,(5x+1)$

$h'(x) = 3\sqrt{x}\,(5) + (5x+1)\frac{3}{2\sqrt{x}}$

$h'(x) = 15\sqrt{x} + \frac{15x+3}{2\sqrt{x}}$

$h'(x) = 15\sqrt{x} + \frac{15}{2}\sqrt{x} + \frac{3}{2\sqrt{x}}$

2. $f(x) = (x^4-2x^3-7)(3x^2-5x)$

$f'(x) = (x^4-2x^3-7)(6x-5) + (3x^2-5x)(4x^3-6x^2)$

$f'(x) = 6x^5 - 12x^4 - 42x - 5x^4 + 10x^3 + 35 + 12x^5 - 18x^4 - 20x^4 + 30x^3$

$f'(x) = 18x^5 - 55x^4 + 40x^3 - 42x + 35$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

In words: $\frac{\text{lo dhi} - \text{hi dlo}}{(\text{lo})^2}$

3. $f(x) = \frac{x^2-3x}{x-1}$ hi / lo

$f'(x) = \frac{(x-1)(2x-3) - (x^2-3x)(1)}{(x-1)^2}$

$f'(x) = \frac{2x^2-5x+3-x^2+3x}{(x-1)^2}$

$f'(x) = \frac{x^2-2x+3}{(x-1)^2}$

4. Find the equation of the tangent line of $f(x) = \frac{3x^2-2}{4x^3+1}$ at $x = 1$

$f'(x) = \frac{(4x^3+1)(6x) - (3x^2-2)(12x^2)}{(4x^3+1)^2}$

$f'(x) = \frac{24x^3 + 6x - 36x^4 + 24x^2}{(4x^3+1)^2}$

$f'(1) = \frac{24+6-36+24}{25}$

$f'(1) = \frac{18}{25}$ m

$f(1) = \frac{3-2}{4+1} = \frac{1}{5}$

$(1, 1/5)$ x_1 y_1

$y - 1/5 = 18/25(x-1)$

4.3 Particle motion

Answer the following questions for each position function $s(t)$ in meters where t is seconds if a particle is moving along the x-axis.

$s(t) = t^3 - 6t^2 + 9t$

a. Find the velocity of time.
$v(t) = s'(t)$
$v(t) = 3t^2 - 12t + 9$

b. What is the velocity after 2 seconds?
$v(2) = 3(2)^2 - 12(2) + 9$ $\quad v(2) = -3\ m/sec$

c. Find the acceleration as a function of time t.
$a(t) = v'(t) = s''(t)$
$a(t) = 6t - 12$

d. Find the acceleration at $t = 3$ seconds.
$a(3) = 6(3) - 12 = 18 - 12$
$a(3) = 6\ m/sec^2$

e. When is the particle at rest?
The particle is at rest when $v(t) = 0$
$3t^2 - 12t + 9 = 0$
$3(t^2 - 4t + 3) = 0$
$3(t-3)(t-1) = 0$
$t = 3\ sec$ & $t = 1\ sec$

f. When is the particle moving forward (or right)?
The particle is moving forward when $v(t) > 0$

++ | - - | +++
1 3

moving forward from $(0,1) \cup (3, \infty)$ seconds

g. What is the displacement on $[0,5]$ seconds?
displacement $= s(5) - s(0)$
$s(5) = 5^3 - 6(5)^2 + 9(5) = 20$ $\quad 20 - 0$
$s(0) = 0^3 - 6(0)^2 + 9(0) = 0$ $\quad$ 20 m

h. Find the total distance traveled on $[0,5]$
$s(0) = 0$ $\quad 0 \to 1 = |4 - 0| = 4$
$s(1) = 1^3 - 6(1) + 9(1) = 4$ $\quad 1 \to 3 = |0 - 4| = 4$
$s(3) = (3)^3 - 6(3)^2 + 9(3) = 0$ $\quad 3 \to 5 = |20 - 0| = 20$ $\quad$ 28 m
$s(5) = 20$

i. Find the velocity when acceleration is $24\ m/sec^2$.
$a(t) = 6t - 12 = 24$ $\quad v(6) = 3(6)^2 - 12(6) + 9$
$6t = 36$ $\quad v(6) = 108 - 72 + 9$ $\quad v(6) = 45\ m/sec$
$t = 6\ sec$

Cont. 4.3

j. Find when the particle is speeding up + slowing down.

Compare the velocity + acceleration.

$a(t) = 0$

$6t - 12 = 0$

$t = 2$

		1		2		3	
$v(t)$	+ + +		−		−		+ + +
$a(t)$	− −		− −		+ +		+ + +
	dif		same		dif		same

The particle is speeding up when the signs of $v(t)$ + $a(t)$ are the same + slowing down when signs are different

speeding up: $(1,2) \cup (3,5)$

slowing down: $(0,1) \cup (2,3)$

4.4 Derivatives of Trigonometric Functions

Trig Derivatives

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}\cot x = -\csc^2 x$

Find the derivative:

1. $f(x) = \sin x \cos x$

$f'(x) = \sin x \cdot -\sin x + \cos x \cdot \cos x$

$f'(x) = -\sin^2 x + \cos^2 x$

$f'(x) = \cos^2 x - \sin^2 x$ ★ Trig identity

$f'(x) = \cos 2x$

2. $f(x) = x^2 \cos x$

$f'(x) = x^2 \cdot -\sin x + \cos x \cdot 2x$

$f'(x) = -x^2 \sin x + 2x \cos x$

You can rewrite w/ trig identities to make this easier. $f(x) = \frac{\cot x}{x}$ then derive

3. $f(x) = \frac{\cos x}{x \sin x}$

$$f'(x) = \frac{(x\sin x)(-\sin x) - \cos x(x \cdot \cos x + \sin x \cdot 1)}{(x \sin x)^2}$$

$$f'(x) = \frac{-x\sin^2 x - x\cos^2 x - \sin x \cos x}{(x\sin x)^2}$$

$$f'(x) = \frac{-x(\sin^2 x + \cos^2 x) - \sin x \cos x}{(x \sin x)^2}$$

$$f'(x) = \frac{-x - \sin x \cos x}{(x \sin x)^2}$$

4. $f(x) = x - 4\csc x + 2\cot x$

$f'(x) = 1 - 4(-\csc x \cot x) + 2(-\csc^2 x)$

$f'(x) = 1 + 4\csc x \cot x - 2\csc^2 x$

4.5 The Chain Rule

The Chain Rule: used to take the derivative of a composition

$$\boxed{[f(g(x))]' = f'(g(x)) \cdot g'(x)}$$

★ derivative of outside times deriv. of inside

Find the derivative:

1. $y = \sec(x^2+\sqrt{2})$
   $y' = \sec(x^2+\sqrt{2})\tan(x^2+\sqrt{2}) \cdot (2x)$
   $y' = 2x\sec(x^2+\sqrt{2})\tan(x^2+\sqrt{2})$

2. $y = 5\cot\left(\frac{2}{x}\right)$
   $y = 5\cot(2x^{-1})$
   $y' = 5 \cdot -\csc^2(2x^{-1}) \cdot (-2x^{-2})$
   $y' = \frac{10}{x^2}\csc^2\left(\frac{2}{x}\right)$

3. $y = \sin(x^2+3x)^2$ ★Double chain
   $y = \sin[(x^2+3x)^2]$ Rewrite
   $y' = \cos[(x^2+3x)^2] \cdot 2(x^2+3x) \cdot (2x+3)$
   $y' = 2(x^2+3x)(2x+3)\cos(x^2+3x)^2$

4. $y = \sin^2(x^2+3x)$ ★Double Chain
   $y = (\sin(x^2+3x))^2$ ← Rewrite
   $y' = 2(\sin(x^2+3x)) \cdot \cos(x^2+3x) \cdot (2x+3)$
   $y' = 2(2x+3)\sin(x^2+3x)\cos(x^2+3x)$

5. $y = (x^3-4)^4$
   $y' = 4(x^3-4)^3 \cdot 3x^2$
   $y' = 12x^2(x^3-4)^3$

4.6 Derivatives of Logarithmic + Exponential Functions

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

let $u=f(x)$ if it isn't ln of just (x)

$$\boxed{\frac{d}{dx}\ln(u) = \frac{1}{f(x)}\cdot du}$$

*meaning 1 over the fxn times deriv. of the fxn.

Find the derivative:

1. $y=\ln(x^3+1)$
   $y'=\frac{1}{x^3+1}\cdot(3x^2)$
   $y'=\frac{3x^2}{x^3+1}$

2. $y=\ln\sqrt{x}$
   $y=\ln x^{1/2}$
   $y=\frac{1}{2}\ln x$ prop. of logs
   $y'=\frac{1}{2}\cdot\frac{1}{x}$
   $y'=\frac{1}{2x}$

3. $y=\frac{\ln x}{x^2}$
   $y'=\frac{x^2\cdot\frac{1}{x}-\ln x\cdot 2x}{(x^2)^2}$
   $y'=\frac{x-2x\ln x}{x^4}$
   $y'=\frac{1-2\ln x}{x^3}$

4. $y=\ln(\ln x)$
   $y'=\frac{1}{\ln x}\cdot\frac{1}{x}$
   $y'=\frac{1}{x\ln x}$

More derivative rules:

*copy problem, times ln base times deriv. of exponent

$\frac{d}{dx}e^x = e^x\cdot 1 = e^x$ $\qquad$ $\frac{d}{dx}e^u = e^u\cdot d(u)$

$\frac{d}{dx}a^x = a^x\cdot\ln a$ $\qquad$ $\frac{d}{dx}a^u = a^u\cdot\ln a\cdot d(u)$

5. $y=e^{2x}$
   $y'=e^{2x}\cdot 2$
   $y'=2e^{2x}$

6. $y=e^{5x^2}$
   $y'=e^{5x^2}\cdot 10x$
   $y'=10xe^{5x^2}$

7. $y=e^{\sin x}$
   $y'=e^{\sin x}\cdot\cos x$
   $y'=\cos x\, e^{\sin x}$

8. $y=e^{\tan x}$
   $y'=e^{\tan x}\cdot\sec^2 x$
   $y'=\sec^2 x\cdot e^{\tan x}$

9. $y=7^{x^2+2x^3}$
   $y'=7^{x^2+2x^3}\cdot\ln 7\cdot(2x+6x^2)$

10. $y=\sin e^{3x}$
    $y'=\cos(e^{3x})\cdot e^{3x}\cdot 3$
    $y'=3e^{3x}\cos e^{3x}$

4.7 Implicit Differentiation

An equation $F(x,y)=0$ can be differentiated directly by chain rule, without solving for y in terms of x.

★ use when you have $x+y$ on both sides

Find the first + second derivative

1. $xy + y^2 = 1$

$x \cdot \frac{dy}{dx} + y \cdot 1 + 2y\frac{dy}{dx} = 0$ ★ Find the deriv. (anytime you take deriv. of y, include $\frac{dy}{dx}$)

$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -y$ Get terms w/ $\frac{dy}{dx}$ on 1 side + factor out

$(x+2y)\frac{dy}{dx} = -y$ Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-y}{x+2y}$$

$$\frac{d^2y}{dx^2} = \frac{(x+2y)\cdot -1\frac{dy}{dx} - (-y)\cdot\left(1+2\frac{dy}{dx}\right)}{(x+2y)^2}$$

Derive $\frac{dy}{dx}$ w/ quotient rule

$$\frac{d^2y}{dx^2} = \frac{\cancel{(x+2y)}\cdot -1\left(\frac{-y}{\cancel{x+2y}}\right) + y\left(1+2\left(\frac{-y}{x+2y}\right)\right)}{(x+2y)^2}$$

★ Now substitute $\frac{dy}{dx}$ into this + simp.

$$\frac{d^2y}{dx^2} = \frac{y + y - \frac{2y^2}{(x+2y)}}{(x+2y)^2} = \frac{2y - \frac{2y^2}{x+2y}}{(x+2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{2y(x+2y) - 2y^2}{x+2y}}{(x+2y)^2} = \frac{2xy + 4y^2 - 2y^2}{(x+2y)^3} = \frac{2xy + 2y^2}{(x+2y)^3}$$

Find the derivative at a point.

2. $2xy + \pi \sin y = 2\pi$ at $(1, \pi/2)$

$2x \cdot 1\frac{dy}{dx} + y \cdot 2 + \pi\cos y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2x + \pi\cos y) = -2y$

$$\frac{dy}{dx} = \frac{-2y}{2x + \pi\cos y}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,\pi/2)} = \frac{-2(\pi/2)}{2(1) + \pi\cos(\pi/2)} = \frac{-\pi}{2+\pi(0)} = -\frac{\pi}{2}$$

4.8 Logarithmic Differentiation

Steps:
- Take the natural log of both sides
- Use properties for logs + natural logs
- Differentiate both sides
- multiply both sides by f(x)

1. $y = x^{3x-4}$ ★ logarithmic diff. to bring x down from exponent

$\ln y = \ln x^{3x-4}$

$\ln y = (3x-4)\ln x$

$\frac{1}{y}\frac{dy}{dx} = (3x-4)\frac{1}{x} + \ln x\,(3)$ Take derivative off both sides

$\frac{dy}{dx} = \left(\frac{3x-4}{x} + 3\ln x\right)y$ Solve for $\frac{dy}{dx}$ + then plug in original problem for y

$\frac{dy}{dx} = \left(\frac{3x-4}{x} + 3\ln x\right)\cdot x^{3x-4}$

2. $y = (2x-1)^5(3x^2+4)^6$ ★ Can use logarithmic diff. to simplify this easier than product + chain

$\ln y = \ln[(2x-1)^5(3x^2+4)^6]$

$\ln y = 5\ln(2x-1) + 6\ln(3x^2+4)$

$\frac{1}{y}\frac{dy}{dx} = \frac{5}{2x-1}\cdot 2 + \frac{6}{3x^2-4}\cdot 6x$

$\frac{dy}{dx} = \left(\frac{10}{2x-1} + \frac{36x}{3x^2-4}\right)y$

$\frac{dy}{dx} = \left(\frac{10}{2x-1} + \frac{36x}{3x^2-4}\right)(2x-1)^5(3x^2+4)^6$

Log Rule Reminders

Product: $\ln ab = \ln a + \ln b$

Power: $\ln b^a = a\cdot\ln b$

Quotient: $\ln\frac{a}{b} = \ln a - \ln b$

4.9

Derivatives of Inverse Trig Functions

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Find the derivative.

1. $y = \tan^{-1}(2x)$

$y' = \frac{1}{1+(2x)^2} \cdot 2$ ☆ Chain Rule!

$y' = \frac{2}{1+4x^2}$

2. $y = \sin^{-1}(x^2)$

$y' = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$

$y' = \frac{2x}{\sqrt{1-x^4}}$

3. $y = \sec^{-1}(x^3)$

$y' = \frac{1}{|x^3|\sqrt{(x^3)^2-1}} \cdot 3x^2$

$y' = \frac{3x^2}{|x^3|\sqrt{x^6-1}}$

Derivative of the Inverse $\boxed{(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}}$

Evaluate:

1. If $f(4)=5$ & $f'(4)=\frac{2}{3}$, find $(f^{-1})'(5)$.

$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))}$ If $f(4)=5$, then $f^{-1}(5)=4$

$(f^{-1})'(5) = \frac{1}{f'(4)}$

$(f^{-1})'(5) = \frac{1}{2/3} = \frac{3}{2}$

2.

x	f(x)	f'(x)
2	3	4
3	35/4	31/4
4	19	13

Find $(f^{-1})'(3)$

$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$

$= \frac{1}{f'(2)} = \frac{1}{4}$

4.10 L'Hopital's Rule

L'Hopital's Rule: Suppose f + g are differentiable & $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that $$\lim_{x\to \pm\infty} f(x) = \pm\infty \quad \lim_{x\to a} g(x) = \pm\infty$$

In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Evaluate the limit.

1. $\lim_{x\to 0} \frac{\sin x}{x}$

must show!

$\lim_{x\to 0} \sin x = 0$

$\lim_{x\to 0} x = 0$

$\therefore$ L'H $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1}$

$\lim_{x\to 0} \cos x = \cos 0 = 1$

2. $\lim_{t\to 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$

$\lim_{t\to 1} 5t^4 - 4t^2 - 1 = 0$

$\lim_{t\to 1} 10 - t - 9t^3 = 0$

$\therefore$ LH: $\lim_{t\to 1} \frac{20t^3 - 8t}{-1 - 27t^2}$

$= \frac{20-8}{-1-27} = \frac{12}{-28} = \frac{-3}{7}$

3. $\lim_{x\to \infty} \frac{e^x}{x^2}$

$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to\infty} x^2 = \infty$

$\therefore$ LH: $\lim_{x\to\infty} \frac{e^x}{2x}$

Still indeterminate so apply L'Hopital's Rule again!

$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to\infty} 2x = \infty$

$\therefore$ LH $\lim_{x\to\infty} \frac{e^x}{2}$

$= \frac{e^\infty}{2} = \infty$

4. $\lim_{h\to 0} \frac{\frac{5}{6+h} - \frac{5}{6}}{h}$

$\lim_{h\to 0} \frac{5}{6+h} - \frac{5}{6} = 0$

$\lim_{h\to 0} h = 0$

$\therefore$ by LH $\lim_{h\to 0} \frac{\frac{5}{6+h} - \frac{5}{6}}{h}$

quotient rule

deriv. of constant

deriv. of h

$\lim_{h\to 0} \frac{\frac{(6+h)(0) - 5(1)}{(6+h)^2} - 0}{1}$

$\lim_{h\to 0} \frac{-5}{(6+h)^2}$

$= \frac{-5}{(6+0)^2} = \frac{-5}{36}$