Additional Review – Applications of Derivatives

1. Let f be a function with a second derivative given by $f''(x) = x^2(x-3)(x-6)$. What are the x-coordinates of the points of inflection of the graph of f?
   - (A) 0 *only*
   - (B) 3 *only*
   - (C) 0 *and* 6 *only*
   - [✓ (D) 3 *and* 6 *only*]
   - (E) 0, 3, *and* 6

f ∪ ∪ ∩ ∪

f'' + | + | − | +

0 3 6

Calculator 2. The derivative of the function f is given by $f'(x) = x^2\cos(x^2)$. How many points of inflection does the graph of f have on the open interval $(-2,2)$?
   - (A) One
   - (B) Two
   - (C) Three
   - (D) Four
   - [✓ (E) Five]

graph f'(x) + see how many extrema are b/n (-2,2)

3. Let f be the function given by $f(x) = 2xe^x$. The graph of f is concave down when
   - [✓ (A) $x < -2$]
   - (B) $x > -2$
   - (C) $x < -1$
   - (D) $x > -1$
   - (E) $x < 0$

$f'(x) = 2xe^x + 2e^x$

$f''(x) = 2xe^x + 2e^x + 2e^x$

$f''(x) = 2e^x(x+2)$

$0 = 2e^x$ $0 = x+2$

DNE $x = -2$

f'' − | +

-2

Calculator 4. The function f has first derivative given by $f'(x) = \frac{\sqrt{x}}{1+x+x^3}$. What is the x-coordiante of the inflection point of the graph of f?
   - (A) 1.008
   - [✓ (B) 0.473]
   - (C) 0
   - (D) −0.278
   - (E) The graph of f has no inflection point.

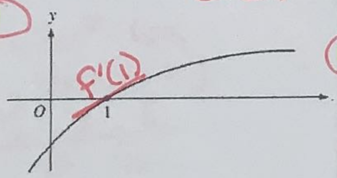
Graph f'(x) + look for extrema. Use Calc (Analyze Graph) to find x-value of max

5. The graph of a twice-differentiable function f is shown in the figure. Which of the following is true?
   - (A) $f(1) < f'(1) < f''(1)$
   - (B) $f(1) < f''(1) < f'(1)$
   - (C) $f'(1) < f(1) < f''(1)$
   - [✓ (D) $f''(1) < f(1) < f'(1)$]
   - (E) $f''(1) < f'(1) < f(1)$

$f(1) = 0$

$f'(1) =$ positive

$f''(1) =$ neg bc conc ↓



6. The function f is given by $f(x) = x^4 + x^2 - 2$. On which of the following intervals is f increasing?
   - (A) $\left(-\frac{1}{\sqrt{2}}, \infty\right)$
   - (B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
   - [✓ (C) $(0, \infty)$]
   - (D) $(-\infty, 0)$
   - (E) $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$

$f'(x) = 4x^3 + 2x$

$0 = 2x(2x^2+1)$ ↖ imag.

$x = 0$

f' − | +

0

f ↘ ↗

7. What are all values of x for which the function f defined by $f(x) = (x^2-3)e^{-x}$ is increasing?
   - (A) There are no such values of x.
   - (B) $x < -1$ and $x > 3$
   - (C) $-3 < x < 1$
   - [✓ (D) $-1 < x < 3$]
   - (E) All values of x

$f'(x) = -(x^2-3)e^{-x} + 2xe^{-x}$

$f'(x) = e^{-x}(-x^2+2x+3)$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, -1$

− | + | −

-1 3

Calculator 8. The graph of the function $y = x^3 + 6x^2 + 7x - 2\cos x$ changes concavity at $x =$
   - (A) −1.58
   - (B) −1.63
   - (C) −1.67
   - [✓ (D) −1.89]
   - (E) −2.33

Solve$\left(\frac{d^2}{dx^2}(f(x)) = 0, x\right)$

Calculator 9. If the derivative of f is given by $f'(x) = e^x - 3x^2$, at which of the following values of x does f have a relative maximum value? (zeros of $f'(x)$)

(A) −0.46
(B) 0.20
[✓ (C) 0.91]
(D) 0.95
(E) 3.73

Look at the graph of $f'(x)$ to see where it changes from + to −

10. At what value of x does the graph of $y = \frac{1}{x^2} - \frac{1}{x^3}$ have a point of inflection?

(A) 0
(B) 1
[✓ (C) 2]
(D) 3
(E) At no value of x

$x=0$ DNE in $f(x)$

$y = x^{-2} - x^{-3}$
$y' = \frac{-2}{x^3} + \frac{3}{x^4}$
$y'' = \frac{6}{x^4} - \frac{12}{x^5}$
$\frac{12}{x^5} = \frac{6}{x^4}$
$6x^5 = 12x^4$
$6x^4(x-2)$
$x = 0, 2$

11. How many critical points does the function $f(x) = (x+2)^5(x-3)^4$ have?

(A) *One*
(B) *Two*
[✓ (C) *Three*]
(D) *Five*
(E) *Nine*

$f'(x) = (x+2)^5 \cdot 4(x-3)^3 + (x-3)^4 \cdot 5(x+2)^4$
$f'(x) = (x+2)^4(x-3)^3(4(x+2) + 5(x-3))$
$0 = (x+2)^4(x-3)^3(9x-7)$
$x = -2, -3, 7/9$

12. The graph of $y = \frac{-5}{x-2}$ is concave downward for all values of x such that

(A) $x < 0$
[✓ (B) $x < 2$]
(C) $x < 5$
(D) $x > 0$
(E) $x > 2$

$y' = \frac{5}{(x-2)^2}$
$y'' = \frac{-10}{(x-2)^3}$

+ | −
2

13. Let f be a polynomial function with degree greater than 2. If $a \neq b$ and $f(a) = f(b) = 1$, which of the following must be true for at least one value of x between a and b ?

I. $f(x) = 0$ II. $f'(x) = 0$
III. $f''(x) = 0$

↑
Rolle's Theorem
$f(a) = f(b)$
$f'(x) = 0$

(A) *None*
(B) *I only*
[✓ (C) *II only*]
(D) *I and II only*
(E) *I, II, and III*

14. The absolute maximum value of $f(x) = x^3 - 3x^2 + 12$ on the closed interval $[-2,4]$ occurs at $x =$

[✓ (A) 4]
(B) 2
(C) 1
(D) 0
(E) −2

$f'(x) = 3x^2 - 6x$
$0 = 3x(x-2)$
$x = 0, 2$

$f(-2) = -8$
$f(0) = 12$
$f(2) = 8$
$f(4) = 28$

15. The function defined by $f(x) = x^3 - 3x^2$ for all real numbers x has a relative maximum at $x =$

(A) −2
[✓ (B) 0]
(C) 1
(D) 2
(E) 4

$f'(x) = 3x^2 - 6x$
$0 = 3x(x-2)$

↗ | ↘ | ↗
0 2

16. If $f(x) = \frac{\ln x}{x}$, for all $x > 0$, which of the following is true?

$f'(x) = \frac{x(\frac{1}{x} - \ln x)}{x^2} = \frac{1 - \ln x}{x^2} = 0$
$1 - \ln x = 0$
$1 = \ln x$
$e^1 = x$
$x = e$

(A) f is increasing for all x greater than 0.
(B) f is increasing for all x greater than 1.
(C) f is decreasing for all x between 0 and 1.
(D) f is decreasing for all x between 1 and e.
[✓ (E) f is decreasing for all x greater than e.]

+ | −
e

Calculator 17. The graph of $y = 5x^4 - x^5$ has a point of inflection at

(A) (0,0) *only*
[✓ (B) (3,162) *only*]
(C) (4,256) *only*
(D) (0,0) *and* (3,162)
(E) (0,0) *and* (4,256)

$y' = 20x^3 - 5x^4$
$y'' = 60x^2 - 20x^3$
$0 = 20x^2(3-x)$
$x = 0, 3$

− | − | +
0 3

$y(3) = 5(3)^4 - (3)^5$
$= 5(81) - 243$
$= 405 - 243$
$y(0) = 162$

18. At $x = 0$, which of the following is true of the function f defined by $f(x) = x^2 + e^{-2x}$?

(A) f is increasing.
[✓ (B) f is decreasing.]
(C) f is discontinuous.
(D) f has a relative minimum.
(E) f has a relative maximum.

$f'(x) = 2x - 2e^{-2x}$
$f'(0) = 2(0) - 2e^{-2(0)}$
$f'(0) = -2$

Free Response Questions

1. Suppose $y = e^{\sin x}$ on the interval $\left[0, \frac{5\pi}{4}\right]$. Identify all absolute maximum and minimum values. Show all work that leads to your answer and give exact answers, not decimal approximations. **(Calculator)**

$y' = \cos x \cdot e^{\sin x}$

$0 = \cos x \quad 0 = e^{\sin x}$

DNE

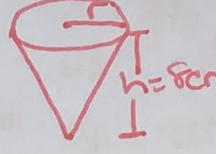
$x = \frac{\pi}{2}$

$f(0) = e^{\sin 0} = 1$

$f(\frac{\pi}{2}) = e^{\sin \pi/2} = e$ ← Abs. max

$f(\frac{5\pi}{4}) = e^{\sin 5\pi/4} = e^{-\sqrt{2}/2} = \frac{1}{e^{\sqrt{2}/2}}$ ← Abs. Min.

2. A coffee filter has the shape of an inverted circular cone with equal base radius and height. Water drains out of the filter at a rate of 10 cm^3/min. At what rate is the height of the water changing when the height of the water is 8 cm? **(Calculator)**



K: $\frac{dV}{dt} = -10 cm^3/min$

F: $\frac{dh}{dt}$

W: h = 8cm, r = 8cm

r = h

$V = \frac{1}{3}\pi r^2 h$

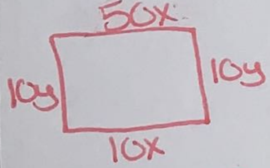
$V = \frac{1}{3}\pi h^3$

$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$

$-10 = \pi(64)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{-5}{32\pi}$ cm/min

3. A gardener wants to build a wall around a 150 ft^2 plot of land. One of the sides is to be a stone wall that will cost \$50 per foot, and the other three sides are to be made of wood that will cost \$10 per foot. What is the minimum cost of the fence (to the nearest whole-number dollar amount)? **(Calculator)**



$A = 150 ft^2$

$150 = xy$

$y = \frac{150}{x}$

$C = 60x + 20y$

$C = 60x + 20(\frac{150}{x})$

$C = 60x + \frac{3000}{x}$

$C' = 60 - \frac{3000}{x^2}$

$\frac{3000}{x^2} = 60$

$60x^2 = 3000$

$x = \sqrt{50}$

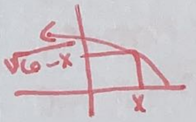
$x = 5\sqrt{2}$

$y = \frac{150}{5\sqrt{2}} = \frac{30}{\sqrt{2}}$

$C = 60(5\sqrt{2}) + 20(\frac{30}{\sqrt{2}})$

C = \$848.53

4. Find the dimensions of the rectangle with largest area that can be inscribed in the region bounded by the curve $y = \sqrt{6 - x}$ in the first quadrant.



$A = x \cdot y$

$A = x\sqrt{6-x}$

$A' = x \cdot \frac{-1}{2\sqrt{6-x}} + \sqrt{6-x} = 0$

$0 = -x + 2(6-x)$

$0 = -x + 12 - 2x$

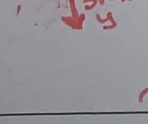
$3x = 12$

$x = 4$

$y = \sqrt{6-4} = \sqrt{2}$

4u x $\sqrt{2}$u

5. A bicyclist is traveling east towards an intersection at the rate of 9 miles per hour. A second bicyclist is traveling south away from the intersection at the rate of 10 miles per hour. What is the rate of change between the bicycles when the first bicycle is 4 miles east of the intersection and the second bicycle is 3 miles south of the intersection?



K: $\frac{dx}{dt} = -9$mph $\frac{dy}{dt} = 10$mph

F: $\frac{dd}{dt}$ W: x = 4mi, y = 3mi

d = 5mi

$x^2 + y^2 = d^2$

$2x\, dx/dt + 2y\, dy/dt = 2d\, dd/dt$

$4(-9) + 3(10) = 5\frac{dd}{dt}$

$\frac{dd}{dt} = \frac{-6}{5}$ mph

6. Find the point on the parabola $y = \sqrt{x}$ that is closest to the point (3,0). $(3,0)$ $(x, \sqrt{x})$

$x_1\ y_1 \quad x_2\ y_2$

$d = \sqrt{(x-3)^2 + (\sqrt{x}-0)^2}$

$d = \sqrt{x^2 - 6x + 9 + x} = \sqrt{x^2 - 5x + 9}$

$d' = \frac{2x-5}{2\sqrt{x^2-5x+9}}$

$0 = 2x - 5$

$x = 5/2$

$y = \sqrt{\frac{5}{2}}$

$(\frac{5}{2}, \sqrt{\frac{5}{2}})$

7. A spherical balloon is inflate with gas at the rate of 500 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a.) 30 centimeters. (b.) 60 centimeters.

K: $\frac{dV}{dt} = 500 cm^3/min.$

F: $\frac{dr}{dt}$

W: $r = \overset{a.}{30cm} + \overset{b.}{60} cm$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$500 = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{125}{\pi r^2}$

a. $\frac{dr}{dt} = \frac{125}{\pi(30)^2}$

$\frac{dr}{dt} = \frac{5}{36\pi}$ cm/min

b. $\frac{dr}{dt} = \frac{125}{\pi(60)^2}$

$\frac{dr}{dt} = \frac{5}{144\pi}$ cm/min.

8. A circular oil slick is being formed in such a way that the radius of the slick is increasing at a constant rate of 12 ft/hr. What will be the rate of area increase when the slick has radius 300 ft?

K: $\frac{dr}{dt} = 12 ft/hr$

F: $\frac{dA}{dt}$

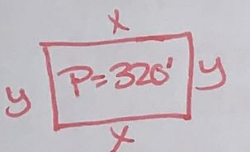
W: $r = 300 ft$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(300)(12) = 7200\pi \ ft^2/hr$

9. A rectangular area is to be enclosed with 320 ft of fence. What dimensions of the rectangle give the maximum area?



$320 = 2x + 2y$

$y = 160 - x$

$A = x \cdot y$

$A = x(160 - x)$

$A = 160x - x^2$

$A' = 160 - 2x$

$2x = 160$

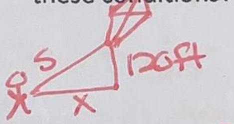
$x = 80 ft$

$y = 160 - 80$

$y = 80 ft$

80 ft x 80 ft

10. A girl is flying a kite. The kite is moving horizontally at a height of 120 ft when 250 ft of string is out and the rate of increase in string length is 2 ft/s. How fast is the kite moving in the horizontal direction for these conditions?



K: $\frac{ds}{dt} = 2 ft/s$

F: $\frac{dx}{dt}$

W: $s = 250$

$y = 120 ft$ → constant

$x^2 + 120^2 = 250^2$

$x = 10\sqrt{481}$

$x^2 + y^2 = s^2$

$2x \frac{dx}{dt} = 2s \frac{ds}{dt}$

$10\sqrt{481} \frac{dx}{dt} = 250(2)$

$\frac{dx}{dt} = \frac{500}{10\sqrt{481}}$

$\frac{dx}{dt} = \frac{50}{\sqrt{481}}$ ft/sec

11. A rectangular pen will be built using 100 feet of fencing. What dimensions will maximize the area?

$100 = 2x + 2y$

$y = 50 - x$

$A = x \cdot y$

$A = x(50 - x)$

$A = 50x - x^2$

$A' = 50 - 2x = 0$

$x = 25 ft$

$y = 50 - 25$

$y = 25 ft$

25 ft x 25 ft

12. Approximate using linearization: ln(1.02)

Let $f(x) = \ln x$ & $x = 1$

$f(1) = \ln(1) = 0$ $\quad (\underset{x_1}{1}, \underset{y_1}{0})$

$f'(x) = \frac{1}{x}$

$f'(1) = \frac{1}{1} = 1 = m_{tan}$

$y = x - 1$

$L(x) = x - 1$

$L(1.02) \approx 1.02 - 1$

$L(1.02) \approx 0.02$

$\ln(1.02) \approx 0.02$

13. Approximate using linearization: $(1.89)^3$

Let $f(x) = x^3$ and $x = 2$

$f(2) = 2^3 = 8 \quad \underset{x_1, y_1}{(2,8)}$

$f'(x) = 3x^2$

$f'(2) = 3(2)^2 = 12 = m_{tan}$

$y - 8 = 12(x-2)$

$y = 12x - 16$

$y(x) = 12x - 16$

$y(1.89) = 12\left(\frac{189}{100}\right) - 16$

$y(1.89) = \frac{567}{25} - \frac{400}{25}$

$(1.89)^3 \approx \frac{167}{25}$

14. Approximate using linearization: $\sqrt[3]{7.999}$

Let $f(x) = \sqrt[3]{x}$ and $x = 8$

$f(8) = \sqrt[3]{8} = 2 \quad (8,2)$

$f'(x) = \frac{1}{3x^{2/3}}$

$f'(8) = \frac{1}{3\sqrt[3]{8^2}} = \frac{1}{12}$

$y - 2 = \frac{1}{12}(x-8)$

$y = \frac{1}{12}x + \frac{4}{3}$

$y(x) = \frac{1}{12}x + \frac{4}{3}$

$y\left(\frac{7999}{1000}\right) = \frac{1}{12}\left(\frac{7999}{1000}\right) + \frac{4}{3}$

$= \frac{23{,}999}{12{,}000}$

15. Approximate using linearization: $\cos(89°)$

Let $f(x) = \cos x$ and $x = 90°$

$f(90) = \cos(90) = 0$

$f'(x) = -\sin x$

$f'(90°) = -\sin(90°) = -1$

$y - 0 = -1(x - 90)$

$y = -x + 90$

$y(x) = -x + 90$

$y(89) = -89 + 90 = 1$

$\cos(89°) \approx 1$

16. Approximate using linearization: $\sqrt{50}$

$f(x) = \sqrt{x}$ & $x = 49$

$f(49) = \sqrt{49} = 7 \quad \underset{x_1, y_1}{(49,7)}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(49) = \frac{1}{2\sqrt{49}} = \frac{1}{14} = m_{tan}$

$y - 7 = \frac{1}{14}(x - 49)$

$y = \frac{1}{14}x - \frac{7}{2} + 7$

$y(x) = \frac{1}{14}x + \frac{7}{2}$

$y(50) = \frac{1}{14}(50) + \frac{7}{2} = \frac{99}{14}$

Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

17. $f(x) = 5 - 12x + 3x^2, \ [1,3]$

$f(x)$ is continuous on $[1,3]$

$f'(x)$ is differentiable on $(1,3)$

$f(1) = 5 - 12(1) + 3(1)^2 = -4$

$f(3) = 5 - 12(3) + 3(3)^2 = -4$

$f(1) = f(3)$

$\therefore \exists c \in [1,3]$ s.t. $f'(c) = 0$

$f'(c) = -12 + 6c$

$0 = -12 + 6c$

$c = 2$

18. $f(x) = x^3 - x^2 - 6x + 2, \ [0,3]$

$f(x)$ is cont. on $[0,3]$

$f'(x)$ is diff. on $(0,3)$

$f(0) = 2$

$f(3) = 27 - 9 - 18 + 2 = 2$ so $f(0) = f(3)$

$\therefore \exists c \in [0,3]$ s.t. $f'(c) = 0$

$f'(c) = 3c^2 - 2c - 6 = 0$

$\frac{2 \pm \sqrt{(4) - 4(3)(-6)}}{6} = \frac{2 \pm \sqrt{76}}{6} = \frac{2 \pm 2\sqrt{19}}{6}$

$= \frac{1 \pm \sqrt{19}}{3}$

$c = \frac{1 + \sqrt{19}}{3}$

19. $f(x)=\sqrt{x}-\frac{1}{3}x$, $[0,9]$

f(x) is cont. on [0,9]
f(x) is diff. on (0,9)
f(0)=0 + f(9)=0 so f(0)=f(9)
$\therefore \exists c \in [0,9]$ s.t. $f'(c)=0$

$f'(c)=\frac{1}{2\sqrt{c}}-\frac{1}{3}=0$

$\frac{1}{2\sqrt{c}}=\frac{1}{3}$

$2\sqrt{c}=3$

$\sqrt{c}=\frac{3}{2}$

$c=9/4$

20. $f(x)=\cos(2x)$, $\left[\frac{\pi}{8},\frac{7\pi}{8}\right]$

f(x) is cont. on $[\pi/8, 7\pi/8]$
f(x) is diff. on $(\pi/8, 7\pi/8)$
$f(\pi/8)=\cos(\frac{\pi}{4})=\frac{\sqrt{2}}{2}$
$f(7\pi/8)=\cos(\frac{7\pi}{4})=\frac{\sqrt{2}}{2}$
$f(\pi/8)=f(7\pi/8)$ $\therefore \exists c \in [\pi/8, 7\pi/8]$ s.t. $f'(c)=0$

$f'(c)=-2\sin(2c)=0$

$\sin(2c)=0$

$2c=\sin^{-1}(0)$

$\frac{2c}{2}=\frac{0}{2},\frac{\pi}{2},\frac{2\pi}{2},\frac{3\pi}{2}$

$c=\frac{\pi}{2}$

Verify that the function satisfies the three hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

17. $f(x)=2x^2-3x+1$, $[0,2]$

f(x) is cont. on [0,2]
f'(x) is diff. on (0,2)
$\therefore \exists c \in [0,2]$ s.t. $f'(c)=\frac{f(2)-f(0)}{2-0}$

$m_{sec}=\frac{f(2)-f(0)}{2}=\frac{3-1}{2}=1$

$f'(c)=4c-3$

$4c-3=1$

$4c=4$

$c=1$

18. $f(x)=x^3-3x+2$, $[-2,2]$

f(x) is cont. on [-2,2]
f'(x) is diff. on (-2,2)
$\therefore \exists c \in [-2,2]$ s.t. $f'(c)=\frac{f(2)-f(-2)}{2+2}$

$m_{sec}=\frac{4-0}{4}=1$

$f'(c)=3c^2-3$

$3c^2-3=1$

$3c^2=4$

$c^2=\frac{4}{3}$

$c=\pm\frac{2}{\sqrt{3}}$

19. $f(x)=\ln(x)$, $[1,4]$

f(x) is cont. on [1,4]
f(x) is diff. on (1,4)
$\therefore \exists c \in [1,4]$ s.t. $f'(c)=\frac{f(4)-f(1)}{4-1}$

$m_{sec}=\frac{\ln 4-\ln 1}{3}=\frac{\ln 4}{3}$

$f'(c)=\frac{1}{c}$

$\frac{\ln 4}{3}=\frac{1}{c}$

$3=\ln 4c$

$c=\frac{3}{\ln 4}$

20. $f(x)=\frac{1}{x}$, $[1,3]$

f(x) is cont. on [1,3]
f'(x) is diff. on (1,3)
$\therefore \exists c \in [1,3]$ s.t. $f'(c)=\frac{f(3)-f(1)}{3-1}$

$m_{sec}=\frac{\frac{1}{3}-1}{2}=\frac{\frac{-2}{3}}{2}=\frac{-2}{6}=-\frac{1}{3}$

$f'(c)=\frac{-1}{c^2}$

$-\frac{1}{c^2}=-\frac{1}{3}$

$c^2=3$

$c=\sqrt{3}$