Unit 7

Integration Applications

- Notes and some practice are included
- Homework will be assigned on a daily basis

Topics Covered:

- ❖ Average Function Value & Mean Value Theorem
- ❖ Area Between 2 Curves
- ❖ Volume by Known Cross Sections
- ❖ Volumes of Revolutions – Disk & Washer Method

Quiz is ____________________

Test is ____________________

Name: Bonanni

F19

$\frac{1}{b-a}\int_a^b f(x)\,dx$

Average Function Value/Mean Value Theorem

For each problem, find the average value of the function over the given interval.

1. $f(x) = -x^2 - 2x + 5;\ [-4, 0]$ (a b)

$\frac{1}{0--4}\int_{-4}^{0} -x^2-2x+5\,dx$

$\frac{1}{4}\left(-\frac{x^3}{3}-x^2+5x\Big|_{-4}^{0}\right)$

$\frac{1}{4}\left[(0-0+0)-\left(\frac{64}{3}-16-20\right)\right]$

$\frac{1}{4}\cdot\frac{44}{3} = \frac{11}{3}$

2. $f(x) = -x^4 + 2x^2 + 4;\ [-2, 1]$

$\frac{1}{1--2}\int_{-2}^{1} -x^4+2x^2+4\,dx$

$\frac{1}{3}\left(-\frac{x^5}{5}+\frac{2x^3}{3}+4x\Big|_{-2}^{1}\right)$

$\frac{1}{3}\left[\left(-\frac{1}{5}+\frac{2}{3}+4\right)-\left(\frac{32}{5}-\frac{16}{3}-8\right)\right]$

$\frac{1}{3}\cdot\frac{57}{5} = \frac{19}{5}$

3. $f(x) = 4 - x^2;\ [-2, 2]$

$\frac{1}{2--2}\int_{-2}^{2} 4-x^2\,dx$

$\frac{1}{4}\left(4x-\frac{x^3}{3}\Big|_{-2}^{2}\right)$

$\frac{1}{4}\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right]$

$\frac{1}{4}\cdot\frac{32}{3} = \frac{8}{3}$

4. $f(x) = \dfrac{x^2+5}{x};\ [1, 2]$

$\frac{1}{2-1}\int_{1}^{2} x+\frac{5}{x}\,dx$

$\frac{x^2}{2}+5\ln|x|\Big|_1^2$

$(\cancel{4}+5\ln 2)-(\tfrac{1}{2}+5\ln 1)$

$\frac{3}{2}+5\ln 2$

5. $f(x) = \sin x;\ [0, \pi]$

$\frac{1}{\pi-0}\int_0^{\pi}\sin x\,dx$

$\frac{1}{\pi}\left(-\cos x\Big|_0^{\pi}\right)$

$\frac{1}{\pi}\left(-\cos\pi--\cos 0\right)$

$\frac{1}{\pi}(1+1) = \frac{2}{\pi}$

6. $f(x) = \cos x;\ \left[0, \frac{\pi}{2}\right]$

$\frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2}\cos x\,dx$

$\frac{2}{\pi}\left(\sin x\Big|_0^{\pi/2}\right)$

$\frac{2}{\pi}\left(\sin\frac{\pi}{2}-\sin 0\right)$

$\frac{2}{\pi}(1-0) = \frac{2}{\pi}$

$$(b-a)f(c) = \int_a^b f(x)\,dx$$

each problem, find the values of c that satisfy the Mean Value Theorem for Integrals.

$f(x) = -x + 2;\ [-2, 2]$

$(2--2)(-c+2) = \int_{-2}^{2} -x+2\,dx$

$4(-c+2) = -\frac{x^2}{2} + 2x \Big|_{-2}^{2}$

$-4c+8 = (-2+4)-(-2-4)$

$-4c+8 = 8$

$c = 0$

8. $f(x) = \frac{4}{x^2};\ [-4, -2]$

$(-2--4)\left(\frac{4}{c^2}\right) = \int_{-4}^{-2} \frac{4}{x^2}\,dx$

$\frac{8}{c^2} = \frac{-4}{x}\Big|_{-4}^{-2}$

$\frac{8}{c^2} = 2-1$

$\frac{8}{c^2} = 1$

$c^2 = 8$

$c = \pm 2\sqrt{2}$

$c = -2\sqrt{2}$

($+2\sqrt{2}$ isn't in the interval)

9. $f(x) = 4\sqrt{x};\ [0,4]$

$(4-0)4\sqrt{c} = \int_0^4 4\sqrt{x}\,dx$

$16\sqrt{c} = \frac{2 \cdot 4}{3}x^{3/2}\Big|_0^4$

$16\sqrt{c} = \frac{8\sqrt{4}^3}{3} - \frac{8\sqrt{0}^3}{3}$

$16\sqrt{c} = \frac{64}{3}$

$\sqrt{c} = \frac{64}{48}$ or $\frac{4}{3}$

$c = \frac{16}{9}$

10. $f(x) = -3(2x-6)^{\frac{1}{2}};\ [3,5]$

$(5-3)\left[-3(2c-6)^{1/2}\right] = \int_3^5 -3(2x-6)^{1/2}\,dx$

$-6(2c-6)^{1/2} = \frac{-3(2x-6)^{3/2} \cdot 2}{3 \cdot 2}\Big|_3^5$

$-6\sqrt{2c-6} = -\sqrt{2x-6}^{\,3}\Big|_3^5$

$-6\sqrt{2c-6} = -8-0$

$\sqrt{2c-6} = \frac{4}{3}$

$2c-6 = \frac{16}{9}$

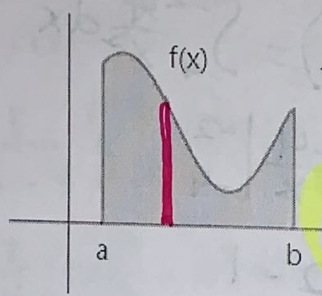
$2c = \frac{70}{9}$

$c = \frac{35}{9}$

Area Between Curves Introduction Worksheet

Draw the arbitrary rectangle and set up the integral to find the area for the shaded region.

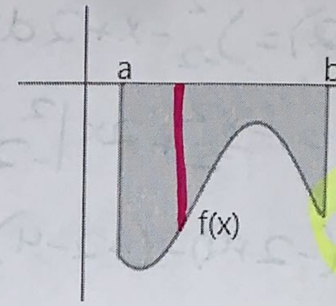
1.



$\int_a^b f(x) - 0 \, dx$

$\int_a^b f(x) \, dx$

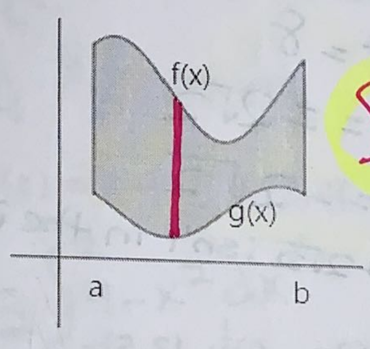
2.



$\int_a^b 0 - f(x) \, dx$

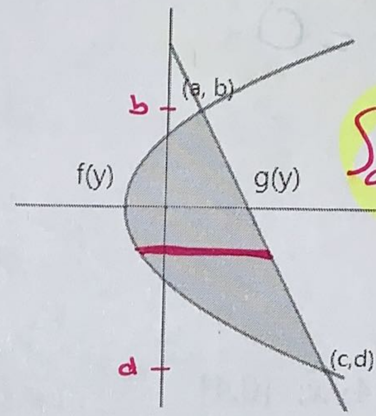
$-\int_a^b f(x) \, dx$

3.



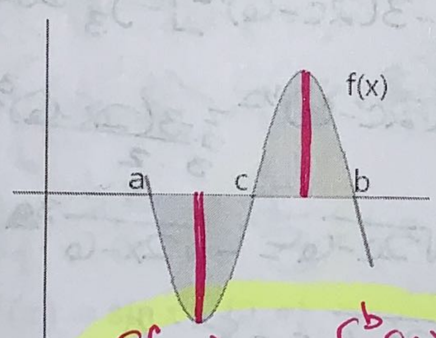
$\int_a^b f(x) - g(x) \, dx$

4.



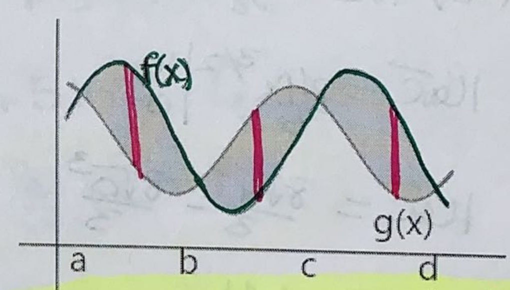
$\int_d^b g(y) - f(y) \, dy$

5.



$-\int_a^c f(x) \, dx + \int_c^b f(x) \, dx$

6.

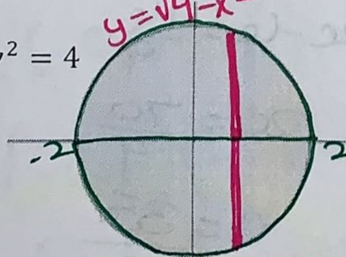


$\int_a^b f(x) - g(x) \, dx + \int_b^c g(x) - f(x) \, dx + \int_c^d f(x) - g(x) \, dx$

7. $x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4 - x^2}$

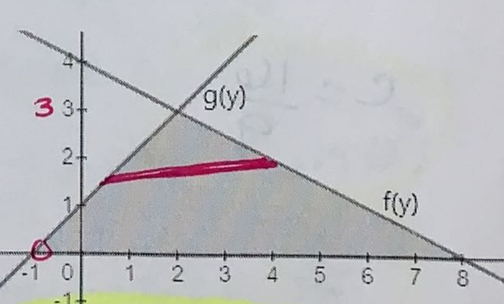


top - bottom

$\int_{-2}^{2} \sqrt{4-x^2} - -\sqrt{4-x^2} \, dx$

$\int_{-2}^{2} 2\sqrt{4-x^2} \, dx$

8.

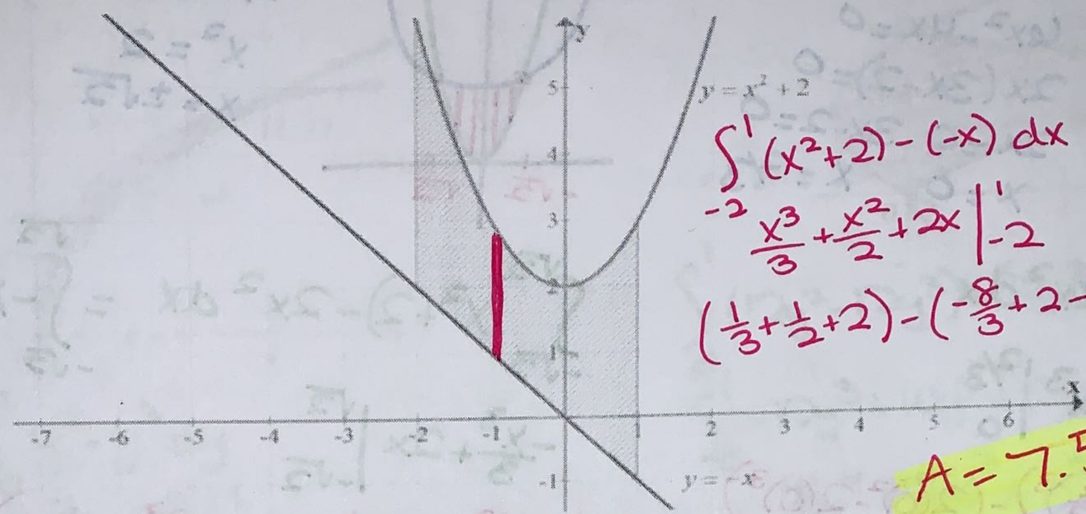


$\int_0^3 f(y) - g(y) \, dy$

Area between Two Curves

Compute the area of the shaded region.

1.



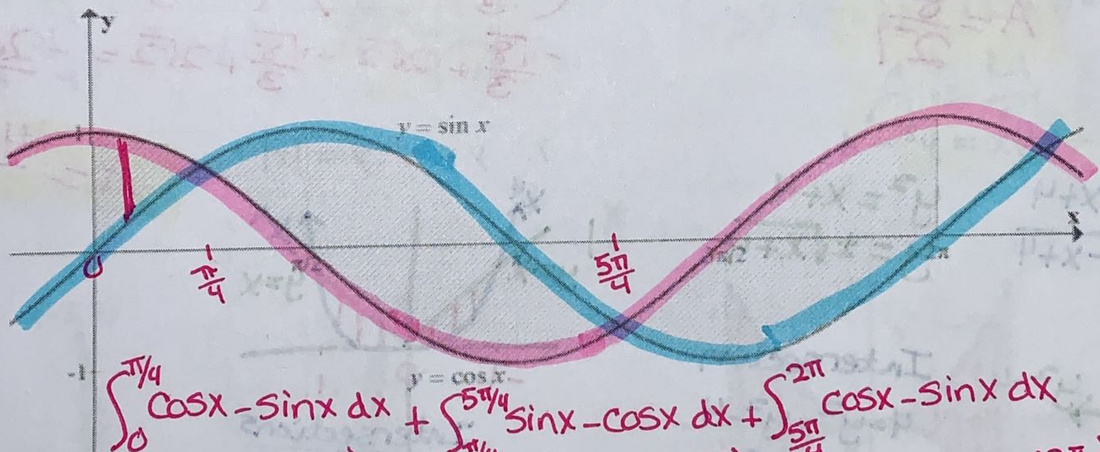
$\int_{-2}^{1} (x^2+2)-(-x)\,dx$

$\frac{x^3}{3}+\frac{x^2}{2}+2x \Big|_{-2}^{1}$

$\left(\frac{1}{3}+\frac{1}{2}+2\right)-\left(-\frac{8}{3}+2-4\right)$

$A = 7.5$

2.

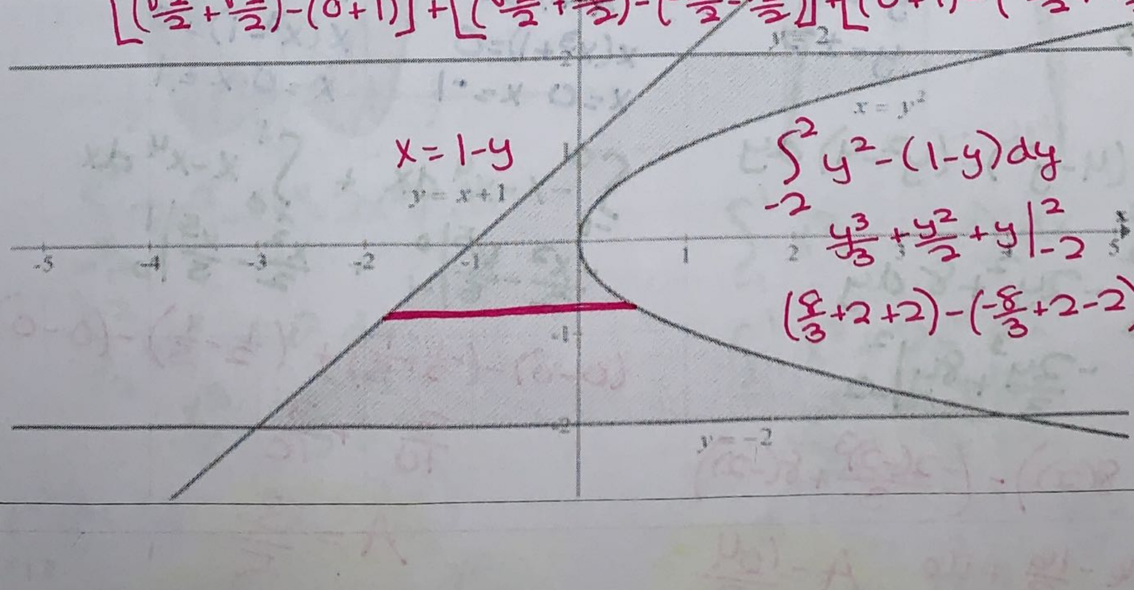


$\int_{0}^{\pi/4} \cos x - \sin x\,dx + \int_{\pi/4}^{5\pi/4} \sin x - \cos x\,dx + \int_{5\pi/4}^{2\pi} \cos x - \sin x\,dx$

$\left(\sin x + \cos x \Big|_{0}^{\pi/4}\right) + \left(-\cos x - \sin x \Big|_{\pi/4}^{5\pi/4}\right) + \left(\sin x + \cos x \Big|_{5\pi/4}^{2\pi}\right)$

$\left[\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)-(0+1)\right]+\left[\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)-\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)\right]+\left[(0+1)-\left(-\frac{\sqrt{2}}{2}+-\frac{\sqrt{2}}{2}\right)\right] = 4\sqrt{2}$

3.



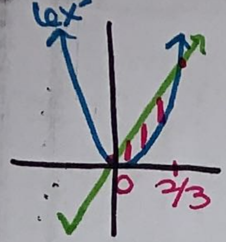
$\int_{-2}^{2} y^2-(1-y)\,dy$

$\frac{y^3}{3}+\frac{y^2}{2}+y \Big|_{-2}^{2}$

$\left(\frac{8}{3}+2+2\right)-\left(-\frac{8}{3}+2-2\right) = \frac{28}{3}$

Compute the area of the region enclosed by the given curves.

4. $y = 4x,\ y = 6x^2$



$6x^2 = 4x$

$6x^2 - 4x = 0$

$2x(3x-2) = 0$

$2x = 0 \quad 3x - 2 = 0$

$x = 0 \quad x = 2/3$

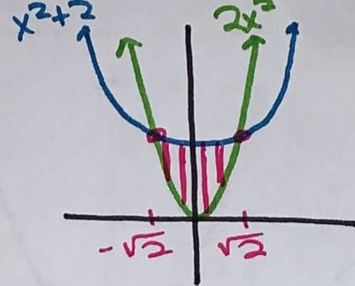
$\int_0^{2/3} 4x - 6x^2\,dx$

$2x^2 - 2x^3 \Big|_0^{2/3}$

$\left(2\left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)^3\right) - \left(2(0)^2 - 2(0)^3\right)$

$\frac{8}{9} - \frac{16}{27} \quad A = \frac{8}{27}$

5. $y = 2x^2, y = x^2 + 2$



$2x^2 = x^2 + 2$

$x^2 = 2$

$x = \pm\sqrt{2}$

$\int_{-\sqrt{2}}^{\sqrt{2}} (x^2+2) - 2x^2\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} -x^2 + 2\,dx$

$-\frac{x^3}{3} + 2x \Big|_{-\sqrt{2}}^{\sqrt{2}}$

$\left(-\frac{(\sqrt{2})^3}{3} + 2\sqrt{2}\right) - \left(-\frac{(-\sqrt{2})^3}{3} + -2\sqrt{2}\right)$

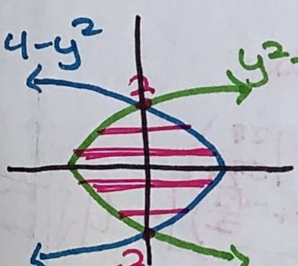
$-\frac{\sqrt{8}}{3} + 2\sqrt{2} - \frac{\sqrt{8}}{3} + 2\sqrt{2} = -\frac{2\sqrt{8}}{3} + 4\sqrt{2}$

$A = \frac{-4\sqrt{2}}{3} + 4\sqrt{3}$

6. $x = 4 - y^2, x = y^2 - 4$

$y^2 = -x + 4 \qquad y^2 = x + 4$

$y = \pm\sqrt{-x+4} \qquad y = \pm\sqrt{x+4}$



Intersection

$4 - y^2 = y^2 - 4$

$-2y^2 = -8$

$y^2 = 4$

$y = \pm 2$

$\int_{-2}^{2} (4-y^2) - (y^2-4)\,dy$

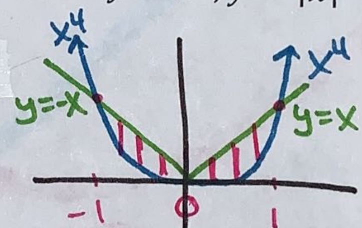
$\int_{-2}^{2} -2y^2 + 8\,dy$

$-\frac{2y^3}{3} + 8y \Big|_{-2}^{2}$

$\left(-\frac{2(2)^3}{3} + 8(2)\right) - \left(-\frac{2(-2)^3}{3} + 8(-2)\right)$

$-\frac{16}{3} + 16 - \frac{16}{3} + 16 \quad A = \frac{64}{3}$

7. $y = x^4, y = |x|$



Intersections

$x^4 = -x \qquad x^4 = x$

$x^4 + x = 0 \qquad x^4 - x = 0$

$x(x^3+1) = 0 \qquad x(x^3-1) = 0$

$x = 0 \quad x = -1 \qquad x = 0 \quad x = 1$

$\int_{-1}^{0} -x - x^4\,dx + \int_0^1 x - x^4\,dx$

$-\frac{x^2}{2} - \frac{x^5}{5}\Big|_{-1}^{0} + \frac{x^2}{2} - \frac{x^5}{5}\Big|_0^1$

$(0-0) - \left(-\frac{1}{2} + \frac{1}{5}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) - (0-0)$

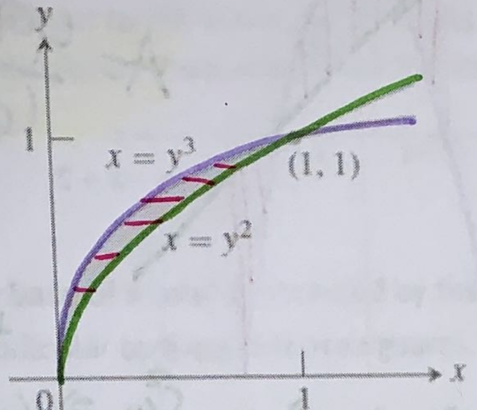
$\frac{3}{10} + \frac{3}{10}$

$A = \frac{3}{5}$

Area between Curves 2

ind the area of the shaded region analytically.

1.



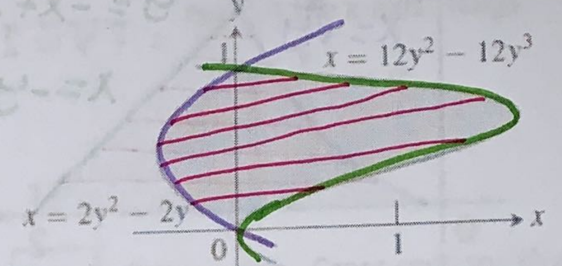
$\int_0^1 y^2 - y^3\,dy$

$\frac{y^3}{3} - \frac{y^4}{4}\Big|_0^1$

$\left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0)$

$A = \frac{1}{12}$

2.



$\int_0^1 (12y^2 - 12y^3) - (2y^2 - 2y)\,dy$

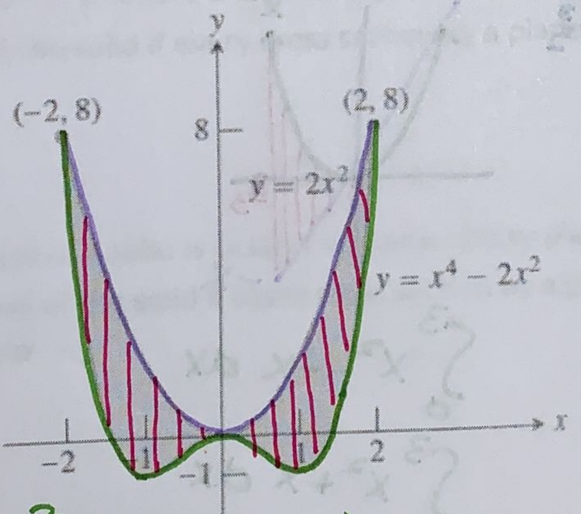
$\int_0^1 -12y^3 + 10y^2 + 2y\,dy$

$-3y^4 + \frac{10y^3}{3} + y^2\Big|_0^1$

$\left(-3 + \frac{10}{3} + 1\right) - (0 + 0 + 0)$

$A = \frac{4}{3}$

3.



$\int_{-2}^{2} 2x^2 - (x^4 - 2x^2)\,dx$

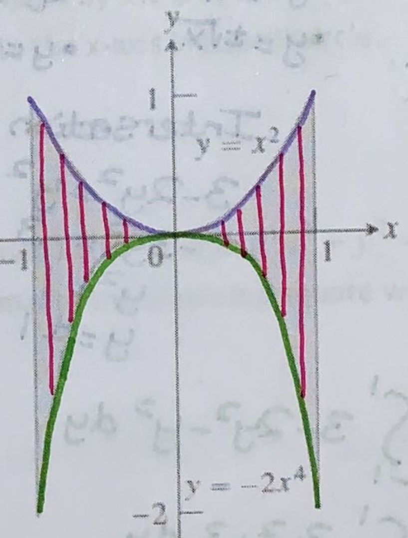
$\int_{-2}^{2} -x^4 + 4x^2\,dx$

$-\frac{x^5}{5} + \frac{4x^3}{3}\Big|_{-2}^{2}$

$\left(-\frac{32}{5} + \frac{32}{3}\right) - \left(\frac{32}{5} - \frac{32}{3}\right)$

$A = \frac{128}{15}$

4.



$\int_{-1}^{1} x^2 - (-2x^4)\,dx$

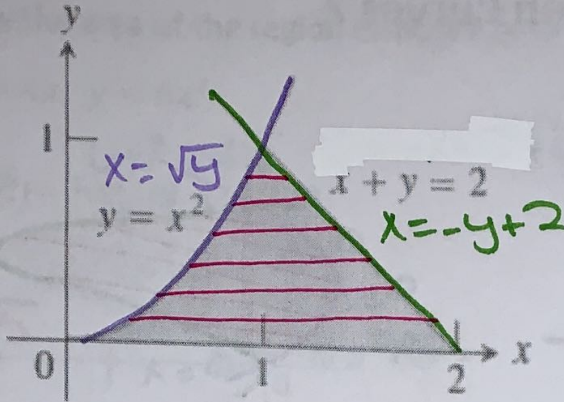
$\int_{-1}^{1} x^2 + 2x^4\,dx$

$\frac{x^3}{3} + \frac{2x^5}{5}\Big|_{-1}^{1}$

$\left(\frac{1}{3} + \frac{2}{5}\right) - \left(-\frac{1}{3} - \frac{2}{5}\right)$

$A = \frac{22}{15}$

5.



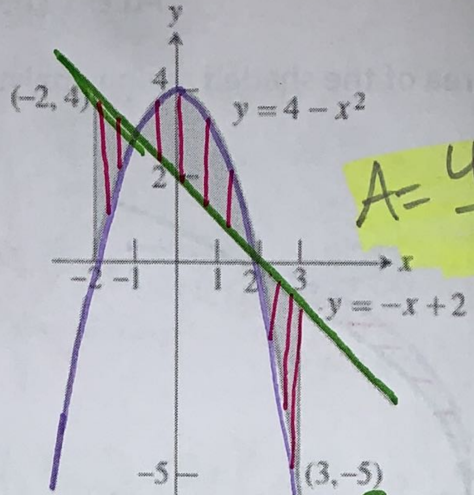
$\int_0^1 -y+2-\sqrt{y}\,dy$

$-\frac{y^2}{2}+2y-\frac{2y^{3/2}}{3}\Big|_0^1$

$(-\frac{1}{2}+2-\frac{2}{3})-(0+0+0)$

$A=5/6$

6.



$A=\frac{49}{6}$

$\int_{-2}^{-1}(-x+2)-(4-x^2)\,dx+\int_{-1}^{2}(4-x^2)-(-x+2)\,dx+\int_2^3(-x+2)-(4-x^2)$

$\int_{-2}^{-1}x^2-x-2+\int_{-1}^{2}-x^2+x+2+\int_2^3 x^2-x-2$

$(\frac{x^3}{3}-\frac{x^2}{2}-2x\Big|_{-2}^{-1})+(-\frac{x^3}{3}+\frac{x^2}{2}+2x\Big|_{-1}^{2})+(\frac{x^3}{3}-\frac{x^2}{2}-2x\Big|_2^3)$

$(-\frac{1}{3}-\frac{1}{2}+2)-(-\frac{8}{3}-2+4)+(-\frac{8}{3}+2+4)-(\frac{1}{3}+\frac{1}{2}-2)+(9-\frac{9}{2}-6)-(\frac{8}{3}-2-4)$

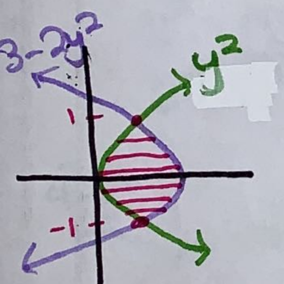
7. Find the area of the region(s) enclosed by the graphs of $x-y^2=0$ and $x+2y^2=3$

$y^2=x$

$y=\pm\sqrt{x}$

$2y^2=-x+3$

$y=\pm\sqrt{-\frac{x}{2}+\frac{3}{2}}$



Intersection

$3-2y^2=y^2$

$-3y^2=-3$

$y^2=1$

$y=\pm 1$

$\int_{-1}^{1}3-2y^2-y^2\,dy$

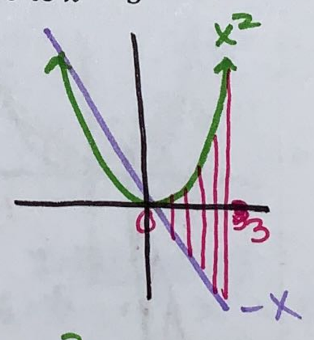
$\int_{-1}^{1}-3y^2+3\,dy$

$-y^3+3y\Big|_{-1}^{1}$

$(-1+3)-(1-3)$

$A=4$

8. Find the area of the region(s) enclosed by the graphs of $y=x^2$ and $y=-x$ from $x=0$ to $x=3$



$\int_0^3 x^2--x\,dx$

$\int_0^3 x^2+x\,dx$

$\frac{x^3}{3}+\frac{x^2}{2}\Big|_0^3$

$(9+\frac{9}{2})-(0+0)$

$A=13.5$

Volumes with Cross Sections

1. The base of a solid is bounded by $y = 2 - x$, the x-axis, and the y-axis. Cross sections that are perpendicular to the x-axis are isosceles right triangles with the right angle on the x-axis. (Legs perpendicular to the x-axis). Find the volume.

$b = 2 - x$

$A = \frac{1}{2}(2-x)^2$

$A = \frac{1}{2}(4 - 4x + x^2)$

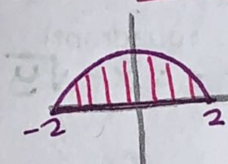
$\int_0^2 2 - 2x + \frac{x^2}{2}\,dx$

$2x - x^2 + \frac{x^3}{6}\Big|_0^2$

$(4 - 4 + \frac{4}{3}) - (0 - 0 + 0)$

$V = \frac{4}{3}$

2. The base of a solid is bounded by the semi-circle $y = \sqrt{4 - x^2}$ & the x-axis. Cross sections that are perpendicular to the x-axis are squares. Find the volume.



$b = \sqrt{4 - x^2}$

$A = b^2$

$A = (\sqrt{4-x^2})^2 = 4 - x^2$

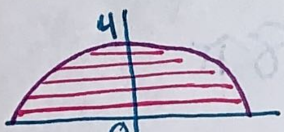
$V = \int_{-2}^{2} 4 - x^2\,dx$

$V = 4x - \frac{x^3}{3}\Big|_{-2}^{2}$

$(8 - \frac{8}{3}) - (-8 + \frac{8}{3})$

$V = \frac{32}{3}$

3. The base of a solid is bounded by $y = \sqrt{16 - x^2}$, the positive x-axis & the positive y-axis. Cross sections that are perpendicular to the y-axis are isosceles right triangles. Find the volume.



$y = \sqrt{16 - x^2}$

$y^2 = 16 - x^2$

$x^2 = 16 - y^2$

$x = \pm\sqrt{16 - y^2}$

$b = \sqrt{16 - y^2}$

$A = \frac{1}{2}(\sqrt{16 - y^2})^2$

$A = 8 - \frac{y^2}{2}$

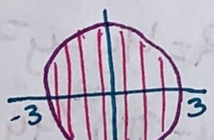
$V = \int_0^4 8 - \frac{y^2}{2}\,dy$

$V = 8y - \frac{y^3}{6}\Big|_0^4$

$(32 - \frac{64}{6}) - (0 - 0)$

$V = \frac{64}{3}$

4. The base of a solid is a circular region in the xy-plane bounded by the graph $x^2 + y^2 = 9$. Find the volume of the solid if every cross section by a plane normal to the x-axis is a semi-circle.



$x^2 + y^2 = 9$

$y = \pm\sqrt{9 - x^2}$

$b = \sqrt{9-x^2} - -\sqrt{9-x^2}$

$b = 2\sqrt{9 - x^2}$

$A = \frac{\pi}{8}(2\sqrt{9-x^2})^2$

$A = \frac{\pi}{8}(4(9 - x^2))$

$A = \frac{\pi}{2}(9 - x^2)$

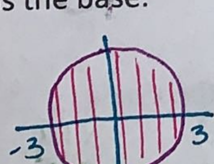
$V = \frac{\pi}{2}\int_{-3}^{3} 9 - x^2\,dx$

$V = \frac{\pi}{2}(9x - \frac{x^3}{3}\Big|_{-3}^{3})$

$V = \frac{\pi}{2}[(27 - 9) - (-27 + 9)]$

$V = 18\pi$

5. The base of a solid is circular region in the xy-plane bounded by the graph of $x^2 + y^2 = 9$. Find the volume of the solid if every cross section by a plane normal to the x-axis is a square with one side as the base.



$b = 2\sqrt{9 - x^2}$

$A = (2\sqrt{9 - x^2})^2$

$A = 4(9 - x^2)$

$A = 36 - 4x^2$

$V = \int_{-3}^{3} 36 - 4x^2\,dx$ or $4\int_{-3}^{3} 9 - x^2\,dx$

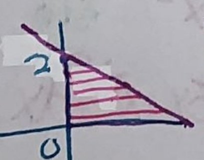
$V = 4(9x - \frac{x^3}{3}\Big|_{-3}^{3})$

$V = 4[(27 - 9) - (-27 + 9)]$

$V = 4 \cdot 36$

$V = 144$

6. The base of a solid is bounded by $y = 2 - \frac{1}{2}x$, the x-axis, and the y-axis. Cross sections that are perpendicular to the y-axis are semi-circles. Find the volume.



$y = 2 - \frac{1}{2}x$

$y - 2 = -\frac{1}{2}x$

$x = -2y + 4$

$b = -2y + 4$

$A = \frac{\pi}{8}(-2y + 4)^2$

$A = \frac{\pi}{8}(4y^2 - 16y + 16)$

$A = \frac{\pi}{2}(y^2 - 4y + 4)$

$V = \int_0^2 \frac{\pi}{2}(y^2 - 4y + 4)\,dy$

$V = \frac{\pi}{2}\int_0^2 y^2 - 4y + 4\,dy$

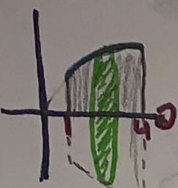
$V = \frac{\pi}{2}(\frac{y^3}{3} - 2y^2 + 4y\Big|_0^2)$

$V = \frac{\pi}{2}((\frac{8}{3} - 8 + 8) - (0 - 0 + 0))$

$V = \frac{4\pi}{3}$

Find the Volumes of Revolution: Disk Method

1. $y = \sqrt{x}, x = 1, x = 4, y = 0$ about the x-axis

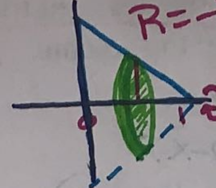


$R = \sqrt{x}$

$V = \pi \int_1^4 (\sqrt{x})^2 dx$

$V = 7.5\pi$ or $\frac{15\pi}{2}$

2. $y = -x + 1, y = 0, x = 0$ about the x-axis

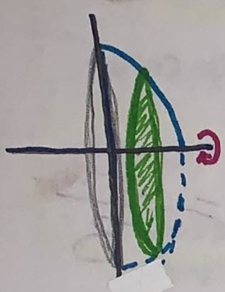


$R = -x+1$

$V = \pi \int_0^1 (-x+1)^2 dx$

$V = \frac{\pi}{3}$

3. $y = 4 - x^2, y = 0, x = 0$, (in the 1st quadrant) about the x-axis



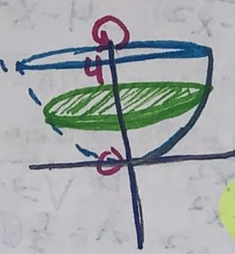
$R = 4 - x^2$

$V = \pi \int_0^2 (4-x^2)^2 dx$

$V = \frac{256\pi}{15}$

4. $y = x^2, x = 0, y = 4$, (in the 1st quadrant) about the y-axis

$y = x^2$ so $x = \sqrt{y}$

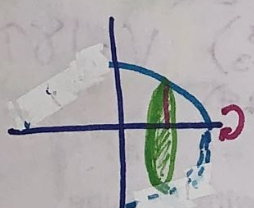


$R = \sqrt{y}$

$V = \pi \int_0^4 (\sqrt{y})^2 dy$

$V = 8\pi$

5. $y = \sqrt{4 - x^2}, y = 0, x = 0$, (in the 1st quadrant) about the x-axis



$R = \sqrt{4-x^2}$

$V = \pi \int_0^2 (\sqrt{4-x^2})^2 dx$

$V = \frac{16\pi}{3}$

6. $x = 4y - y^2, y = 1, x = 0$, about the y-axis

$x = 4y - y^2$

$0 = y(4-y)$

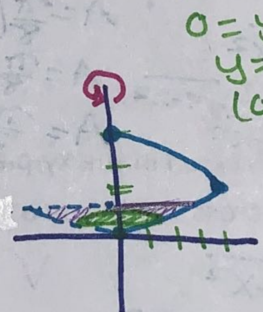
$y=0 \quad y=4$

$(0,0) \quad (0,4)$

$x = 4(2)-(2)^2$

$x = 4$

$(4,2)$

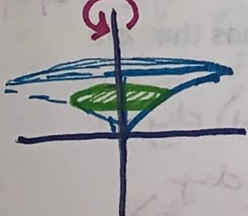


$R = 4y - y^2$

$V = \pi \int_0^1 (4y-y^2)^2 dy$

$V = \frac{53\pi}{15}$

7. $y = x^{\frac{2}{3}}, y = 1, x = 0$, about the y-axis



$y = x^{2/3}$

$x = y^{3/2} \quad R = y^{3/2}$

$V = \pi \int_0^1 (y^{3/2})^2 dy$

$V = \pi \int_0^1 y^3 dy$

$V = \frac{\pi}{4}$

8. $y = 5x - x^2, y = 0$, about the x-axis

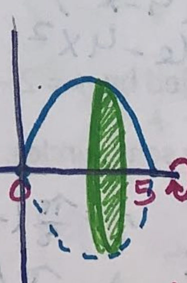
$y = x(5-x)$

$0 = x \quad 0 = 5 - x$

$(0,0) \quad x = 5$

$(5,0)$

x	y
0	0
1	4
2	6
3	6
4	4
5	0



$R = 5x - x^2$

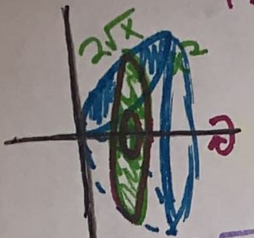
$V = \pi \int_0^5 (5x-x^2)^2 dx$

$V = \frac{625\pi}{6}$

Find the Volumes of Revolution: Washer Method

1. $f(x) = 2\sqrt{x}$ *and* $g(x) = x^2$ about the x-axis

$R = 2\sqrt{x}$ $r = x^2$

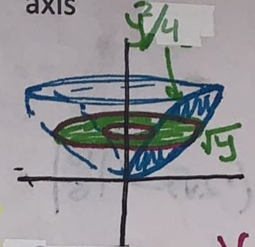


$V = \pi \int_0^{\sqrt[3]{4}} (2\sqrt{x})^2 - (x^2)^2 \, dx$

$V \approx 3.02\pi$ or 9.5

$x^2 = 2\sqrt{x}$ $x(x^3-4)=0$
$x^4 = 4x$ $x=0$ $x=\sqrt[3]{4}$
$x^4 - 4x = 0$

2. $f(x) = 2\sqrt{x}$ *and* $g(x) = x^2$ about the y-axis



$y = 2\sqrt{x}$ $y = x^2$
$y^2 = 4x$ $x = \sqrt{y}$
$x = \frac{y^2}{4}$

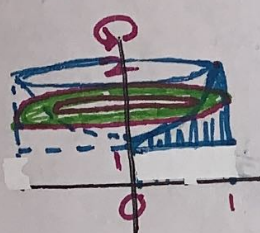
$R = \sqrt{y}$ $r = \frac{y^2}{4}$

$V = \pi \int_0^{\sqrt[3]{16}} (\sqrt{y})^2 - \left(\frac{y^2}{4}\right)^2 dy$

$\frac{y^2}{4} = \sqrt{y}$
$\frac{y^4}{16} = y$ $y^4 - 16y = 0$
$y^4 = 16y$ $y(y^3-16)=0$

$V \approx 1.905\pi$ or 5.984

3. $y = x^2 + 1, y = 0, x = 1, x = 0$ about the y-axis

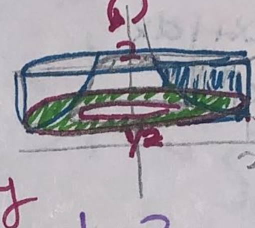


$y = x^2+1$ $R = 1$
$y - 1 = x^2$ $r = \sqrt{y-1}$
$x = \sqrt{y-1}$

$V = \pi \int_1^2 (1)^2 - (\sqrt{y-1})^2 dy$

$V = \frac{\pi}{2}$

4. $y = \frac{1}{x}, y = 2$, *and* $x = 2$ about the y-axis



$y = \frac{1}{x}$ $R = 2$
$xy = 1$ $r = \frac{1}{y}$
$x = \frac{1}{y}$

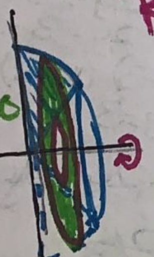
$V = \pi \int_{1/2}^{2} (2)^2 - \left(\frac{1}{y}\right)^2 dy$

$\frac{1}{y} = 2$
$2y = 1$
$y = 1/2$

$V = \frac{9\pi}{2}$

5. $y = x, y = 2 - x^2$, *and* $x = 0$ about the x-axis

$x = 2 - x^2$
$x^2 + x - 2 = 0$
$(x+2)(x-1) = 0$
$x = -2$
$x = 1$

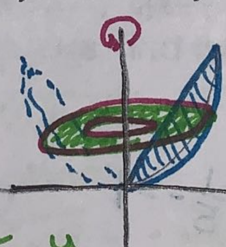


$R = 2 - x^2$ $r = x$

$V = \pi \int_0^1 (2-x^2)^2 - x^2 \, dx$

$V = \frac{38\pi}{15}$

6. $y = x^2$ *and* $y = 2x$, about the y-axis



$y = x^2$ $y = 2x$
$x = \sqrt{y}$ $x = \frac{y}{2}$

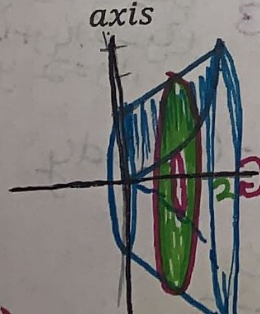
$R = \sqrt{y}$ $r = \frac{y}{2}$

$V = \pi \int_0^4 (\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 dy$

$\sqrt{y} = \frac{y}{2}$
$y = \frac{y^2}{4}$ $y^2 - 4y = 0$
$y^2 = 4y$ $y(y-4) = 0$
$y = 0$ $y = 4$

$V = \frac{8\pi}{3}$

7. $y = x^2$, *and* $y = x + 2$, about the *x-axis*



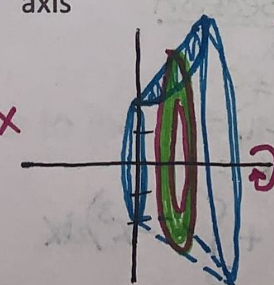
$R = x+2$ $r = x^2$

$V = \pi \int_0^2 (x+2)^2 - (x^2)^2 dx$

$V = \frac{184\pi}{15}$

$x^2 = x+2$
$x^2 - x - 2 = 0$
$(x-2)(x+1) = 0$
$x = 2$ $x = -1$

8. $y = 2x + 2$ *and* $y = x^2 + 2$ about the x-axis



$R = 2x+2$ $r = x^2+2$

$V = \pi \int_0^2 (2x+2)^2 - (x^2+2)^2 \, dx$

$V = \frac{48\pi}{5}$

Unit 7 Integration Applications Review

Find the average value of the function over the interval:

1. $f(x) = \frac{1}{\sqrt{x-1}}$ [5,10]

$\frac{1}{10-5}\int_5^{10} \frac{1}{\sqrt{x-1}}\,dx$

$\frac{1}{5}\int_5^{10} (x-1)^{-1/2}dx = \frac{1}{5}\left(2\sqrt{x-1}\,\Big|_5^{10}\right)$

$\frac{1}{5}(6-4) = 2/5$

2. $f(x) = x^3$ [0,2]

$\frac{1}{2-0}\int_0^2 x^3\,dx$

$\frac{1}{2}\left(\frac{x^4}{4}\Big|_0^2\right) = 2$

Find the value of c guaranteed by the Mean Value Theorem:

3. $f(x) = -2x + 1$ [0,4]

$(4-0)(-2c+1) = \int_0^4 -2x+1\,dx$

$-8c+4 = -x^2+x\,\Big|_0^4$

$-8c+4 = -12$

$-8c = -16$

$c = 2$

4. $f(x) = \frac{2}{x^2}$ [2,4]

$\int 2x^{-2} = \frac{2x^{-1}}{-1} = \frac{-2}{x}$

$(4-2)\frac{2}{c^2} = \int_2^4 \frac{2}{x^2}dx$

$\frac{4}{c^2} = \frac{-2}{x}\Big|_2^4 \quad -\frac{1}{2}--1$

$\frac{4}{c^2} = \frac{1}{2}$

$c^2 = 8$

$c = 2\sqrt{2}$

Determine the area of the bounded region:

5. $y = \frac{1}{x^2}$, $y = 0$, $x = 1$, $x = 5$

$\int_1^5 \frac{1}{x^2}dx$

$-\frac{1}{x}\Big|_1^5 = -\frac{1}{5}--1$

$A = 4/5$

6. $x = y^2 - 2y$, $x = 3$

$0 = y(y-2)$

$y=0 \quad y=2$

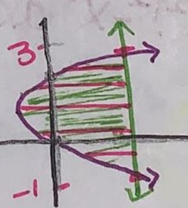
Intersection

$y^2-2y=3$

$y^2-2y-3=0$

right-left $(y-3)(y+1)=0$

$y=3 \quad y=-1$

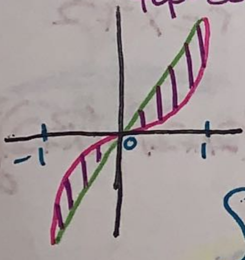


$\int_{-1}^{3} 3-(y^2-2y)\,dy$

$A = 32/3$

7. $y = x$, $y = x^3$

top-bottom



Intersection

$x^3 = x$

$x^3 - x = 0$

$x(x^2-1) = 0$

$x=0 \quad x=\pm 1$

$\int_{-1}^{0}(x^3-x)dx + \int_0^1(x-x^3)dx$

$A = 1/2$

8. $x = y^2 + 1$, $x = y + 3$

$y = \pm\sqrt{x-1} \qquad y = x-3$

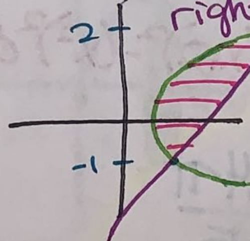
$y^2+1 = y+3$

$y^2-y-2=0$

$(y-2)(y+1)=0$

$y=2 \quad y=-1$

right - left

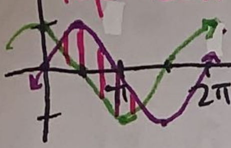


$\int_{-1}^{2}(y+3)-(y^2+1)dy$

$A = \frac{9}{2}$

. $y = \sin x, y = \cos x, \frac{\pi}{4} \le x \le \frac{5\pi}{4}$

top-bottom

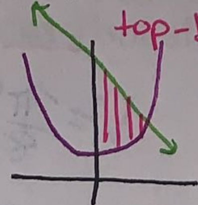


$$\int_{\pi/4}^{5\pi/4} \sin x - \cos x \, dx$$

$A = 2\sqrt{2}$ or 2.83

10. $y = x^2 + 1,\ y = -x + 7, x = 0$

top-bottom



$x^2 + 1 = -x + 7$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

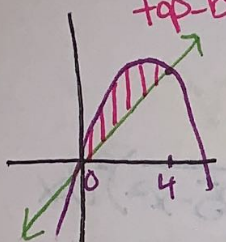
$\cancel{x=-3}\ \ x = 2$

$$\int_0^2 (-x+7) - (x^2+1)\, dx$$

$A = \frac{22}{3}$

11. $y = 5x - x^2,\ y = x$

top-bottom



$5x - x^2 = x$

$0 = x^2 - 4x$

$0 = x(x-4)$

$x = 0,\ x = 4$

$$\int_0^4 (5x - x^2) - (x)\, dx$$

$$\int_0^4 -x^2 + 4x \, dx$$

$A = \frac{32}{3}$

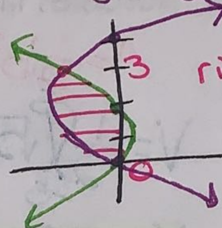
12. $x = y^2 - 4y$ (left), $x = 2y - y^2$ (right)

$0 = y(y-4)$

$y = 0\ \ y = 4$

$0 = y(2-y)$

$y = 0\ \ y = 2$



right-left

$y^2 - 4y = 2y - y^2$

$2y^2 - 6y = 0$

$2y(y-3) = 0$

$y = 0\ \ y = 3$

$$\int_0^3 (2y - y^2) - (y^2 - 4y)\, dy$$

$A = 9$

Find the volume by cross sections:

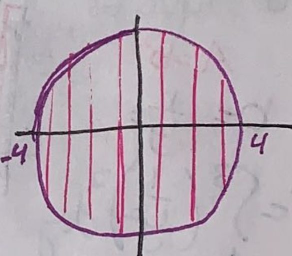
13. The base of a solid is the region enclosed by the circle $x^2 + y^2 = 16$. If cross sections are built up perpendicular to the x-axis, find the volume of the solid created if the cross sections are:

$y = \pm\sqrt{16 - x^2}$

top-bottom

$b = \sqrt{16-x^2} - -\sqrt{16-x^2}$

$b = 2\sqrt{16-x^2}$



a) squares $A = b^2$

$$V = \int_{-4}^{4} (2\sqrt{16-x^2})^2 dx$$

$V = 1024/3$

b) isosceles right triangles set on the hypotenuse $A = \frac{1}{2}b^2$

$$V = \frac{1}{2}\int_{-4}^{4} (2\sqrt{16-x^2})^2 dx$$

$V = 512/3$

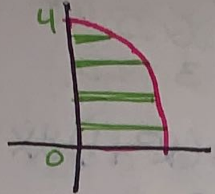
c) What if cross sections are perpendicular to the y-axis and are semi-circles?

$A = \frac{\pi}{8} b^2$

$$V = \frac{\pi}{8}\int_{-4}^{4} (2\sqrt{16-x^2})^2 dx$$

$V = \frac{128\pi}{3}$

14. The base of a solid is the region between $y = 4 - x^2$, $x=0$, $y=0$. If cross sections are perpendicular to the y-axis and are semicircles, find the volume.



$y = 4 - x^2$

$x^2 = 4 - y$

$x = \sqrt{4-y}$

$b = \sqrt{4-y}$

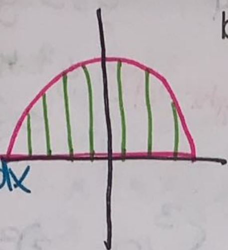
$A = \frac{\pi}{8} b^2$

$V = \frac{\pi}{8} \int_0^4 (\sqrt{4-y})^2 dy$

$V = \frac{\pi}{8} \cdot 8$

$V = \pi$

15. Find the volume of the region generated by $y = \sqrt{25 - x^2}$ and the x-axis. The cross sections are perpendicular to the x-axis:



a. Squares

$A = b^2$

$V = \int_{-5}^{5} (\sqrt{25-x^2})^2 dx$

$V = 500/3$

b. Isosceles triangles

$A = \frac{1}{2} b^2$

$V = \frac{1}{2} \int_{-5}^{5} (\sqrt{25-x^2})^2 dx$

$V = 250/3$

c. Semi Circles

$A = \frac{\pi}{8} b^2$

$V = \frac{\pi}{8} \int_{-5}^{5} (\sqrt{25-x^2})^2 dx$

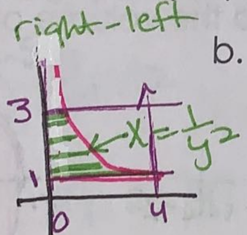
$V = 125\pi/6$

16. Find the volume of the region generated by $y = \frac{1}{\sqrt{x}}$, $x = 0, x = 4, y = 1 \;\&\; y = 3$. The cross sections are perpendicular to the y-axis:

$y = \frac{1}{\sqrt{x}}$ $\quad xy^2 = 1$

$y^2 = \frac{1}{x}$ $\quad x = \frac{1}{y^2}$

right-left



a. Squares

$A = b^2$

$b = \frac{1}{y^2}$

$V = \int_1^3 \left(\frac{1}{y^2}\right)^2 dy$

$V = 26/81$

b. Isosceles triangles

$A = \frac{1}{2} b^2$

$V = \frac{1}{2} \int_1^3 \left(\frac{1}{y^2}\right)^2 dy$

$V = 13/81$

c. Semi Circles

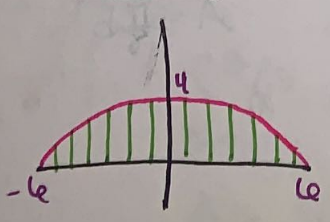
$A = \frac{\pi}{8} b^2$

$V = \frac{\pi}{8} \int_1^3 \left(\frac{1}{y^2}\right) dy$

$V = 13\pi/324$

17. Find the volume of the region generated by $y = -\frac{x^2}{9} + 4$ and $y = 0$. The cross sections are perpendicular to the x-axis. The cross sections are rectangles with a height twice the base.

$h = 2b$



$A_{rectangle} = b \cdot h$

$A = b \cdot 2b$

$A = 2b^2$

$V = 2 \int_{-6}^{6} \left(-\frac{x^2}{9} + 4\right)^2 dx$

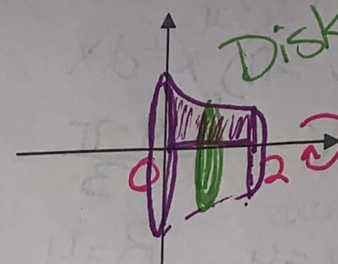
$V = 512/5$

nd the volume of the revolution.

raw the graph, draw the arbitrary cross section, set up the integral, & find the volume.

18. $y = -\sqrt{x} + 3$, y = 0, x = 0 and x = 2

Disk

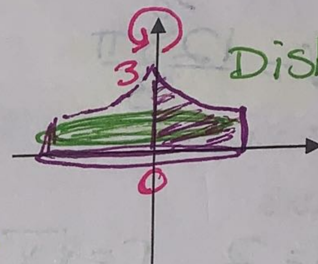


a. about the x axis.

$$V = \pi \int_0^2 (-\sqrt{x}+3)^2 \, dx$$

$V = 8.69\pi$

Disk



b. about the y-axis:

$y = -\sqrt{x} + 3$

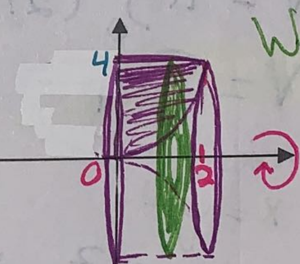
$\sqrt{x} = -y + 3$

$x = (-y+3)^2$

$$V = \pi \int_0^3 ((-y+3)^2)^2 \, dy$$

$V = \frac{243\pi}{5}$

Washer



19. $y = x^2$, $x = 0$, $y = 4$

$R = 4$ $r = x^2$

a. about the x-axis.

$$V = \pi \int_0^2 (4^2) - (x^2)^2 \, dx$$

$V = \frac{128\pi}{5}$

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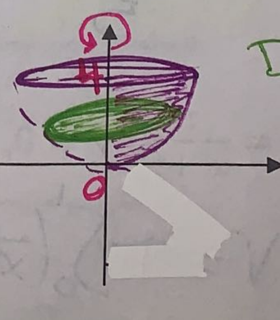

b. about the y-axis.

$y = x^2$

$x = \sqrt{y}$

$$V = \pi \int_0^4 (\sqrt{y})^2 \, dy$$

$V = 8\pi$

Find the volume of the solid generated by revolving the plane region bounded by the indicated equations:

20. $y = x, y = 0, x = 4$

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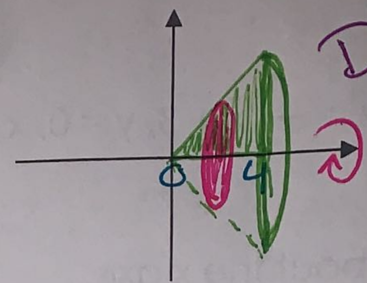


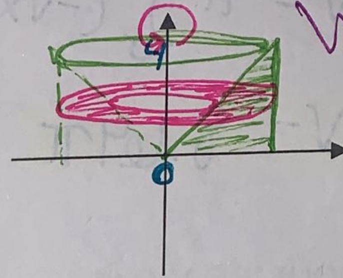
a. x-axis

$r = x$

$V = \pi \int_0^4 x^2\,dx$

$V = \frac{64\pi}{3}$

Washer



b. y-axis

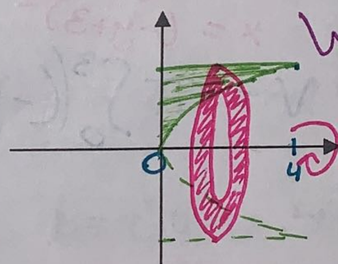
$x = y$ $\quad R = 4$ $\quad r = y$ $\quad$ right − left

$V = \pi \int_0^4 (4)^2 - (y)^2\,dy$

$V = \frac{128\pi}{3}$

21. $y = \sqrt{x}, y = 2, x = 0$

Washer



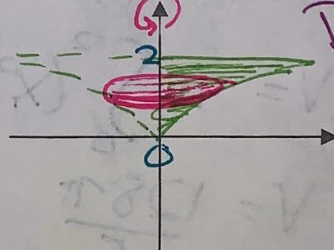
a. x-axis

$R = 2 \quad r = \sqrt{x}$ $\quad$ top − bottom

$V = \pi \int_0^4 (2)^2 - (\sqrt{x})^2\,dx$

$V = 8\pi$

Disk



b. y-axis

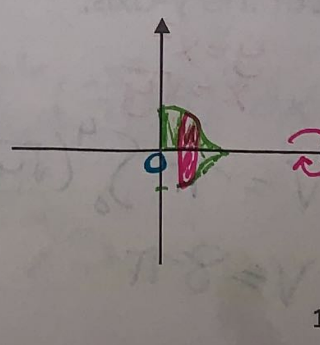
$x = y^2$

$V = \pi \int_0^2 (y^2)^2\,dy$

$V = \frac{32\pi}{5}$

22. $y = \frac{1}{x^4+1}$, $y = 0$, $x = 0, x = 1$ about the x-axis

Disk



$V = \pi \int_0^1 \left(\frac{1}{x^4+1}\right)^2 dx$

$V = .78\pi$