Unit 4B – Applications of Derivatives

- Notes and some practice are included
- Homework will be assigned on a daily basis

Topics Covered:

- Mean Value Theorem
- Particle Motion
- Optimization
- Implicit Differentiation
- Related Rates

Quiz is ____________________

Test is ____________________

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F19

Unit 4B Derivative Applications Review

1. $s(t) = t^3 - 12t^2$ [0,10]

a. What is the velocity function?

$v(t) = 3t^2 - 24t$

b. What is the velocity at $t = 3$ seconds?

$v(3) = 3(3)^2 - 24(3) = -45\text{ m/s}$

c. When is the particle at rest?

$0 = 3t^2 - 24t$

$0 = 3t(t-8)$

$t = 0$ sec

$t = 8$ sec

d. When is the particle moving right? left?

− − − − 0 + +

0   8   10

right: $(8,10)$

left: $(0,8)$

e. What is the acceleration function?

$a(t) = 6t - 24$

f. What is the acceleration at $t = 1$ second?

$a(1) = 6(1) - 24 = -18\text{ m/s}^2$

g. What is the displacement?

$S(0) = 0\text{m}$ $S(10) = -200\text{m}$

200m

h. What is the total distance traveled?

$S(0) = 0$

$S(8) = -256$

$S(10) = -200$

$(0,8) = 256\text{m}$

$(8,10) = +56\text{m}$

312m

i. When is the particle speeding up? Slowing Down?

$0 = 6t - 24$

$t = 4$

v(t): − − | − − | + +   (0, 8, 10)

a(t): − − | + + + +   (0, 4, 10)

speed up: $(0,4) \cup (8,10)$

slow down: $(4,8)$

j. Find the velocity when the acceleration is 0.

$0 = 6t - 24$

$t = 4$ s

$v(4) = 3(4)^2 - 24(4)$

$v(4) = -48\text{ m/s}$

2. $s(t) = -t^3 + 13t^2 - 40t$ [0,6]

a. What is the velocity function?

$v(t) = -3t^2 + 26t - 40$

b. What is the velocity at $t = 3$ seconds?

$v(3) = 11\text{ m/s}$

c. When is the particle at rest?

$0 = -1(3t^2 - 26t + 40)$

$0 = -1(3t - 20)(t - 2)$

$t = 2$ sec

$t = \frac{20}{3} = 6\frac{2}{3}$ not in the interval

d. When is the particle moving right? left?

− − + +

0 2 6

right: $(2,6)$

left: $(0,2)$

e. What is the acceleration function?

$a(t) = -6t + 26$

f. What is the acceleration at $t = 1$ second?

$a(1) = -6(1) + 26$

$= 20\text{ m/s}^2$

g. What is the displacement?

$S(0) = 0$ $S(6) = 12$

$12 - 0 = 12\text{m}$

h. What is the total distance traveled?

$S(2) = -36\text{m}$

$(0,2) = 36$

$(2,6) = +48$

84m

i. When is the particle speeding up? Slowing Down?

v(t): − − | + + + +   (0, 2, 6)

a(t): + + + + | − −   (0, 13/3, 6)

speed up: $(2, \frac{13}{3})$

slow down: $(0,2) \cup (\frac{13}{3}, 6)$

j. Find the velocity when the acceleration is 0.

$0 = -6t + 26$

$t = \frac{26}{6} = \frac{13}{3}$

$v(13/3) = -3(\frac{13}{3})^2 + 26(\frac{13}{3}) - 40$

$49/3\text{ m/s} \approx 16.3\text{ m/s}$

Implicit Differentiation: Find the derivative with respect to x.

3. $x^2 + y^2 = 100$

$2x + 2y\,dy/dt = 0$

$2y\,dy/dt = -2x$

$dy/dt = \frac{-2x}{2y} = \frac{-x}{y}$

4. $xy = x$

$x\frac{dy}{dx} + y = 1$

$x\frac{dy}{dx} = 1 - y$

$\frac{dy}{dx} = \frac{1-y}{x}$

5. $\frac{1}{x} + \frac{1}{y} = 5$

$x^{-1} + y^{-1} = 5$

$-x^{-2} - y^{-2}\,dy/dx = 0$

$-y^2 \cdot -\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x^2} \cdot -y^2$

$\frac{dy}{dx} = \frac{-y^2}{x^2}$

6. $x^3y + y = 1$

$x^3\frac{dy}{dx} + y \cdot 3x^2 + 1\frac{dy}{dx} = 0$

$x^3\frac{dy}{dx} + 1\frac{dy}{dx} = -3x^2y$

$dy/dx\,(x^3+1) = -3x^2y$

$dy/dx = \frac{-3x^2y}{x^3+1}$

7. $\sin(y^2) = 3x$

$\cos(y^2) \cdot 2y\,dy/dx = 3$

$\frac{dy}{dx} = \frac{3}{2y\cos(y^2)}$

8. $\cos(xy) = 2$

$\frac{-\sin(xy)(x\,dy/dx + y)}{-\sin(xy)} = \frac{0}{-\sin(xy)}$

$x\,dy/dx + y = 0$

$x\,dy/dx = -y$

$dy/dx = \frac{-y}{x}$

9. $x^2 + y + xy^3 = 5$

$2x + dy/dx + x \cdot 3y^2\,dy/dx + y^3 = 0$

$dy/dx + 3xy^2\,dy/dx = -2x - y^3$

$dy/dx\,(1 + 3xy^2) = -2x - y^3$

$dy/dx = \frac{-2x - y^3}{1 + 3xy^2}$

Find the equation of the tangent line to the curve given.

10. $x^2y + y = 2$ at $(1, 3)$

$x^2\,dy/dx + y \cdot 2x + dy/dx = 0$

$dy/dx\,(x^2+1) = -2xy$

$dy/dx = \frac{-2xy}{x^2+1}$

$= \frac{-2(1)(3)}{(1)^2+1} = \frac{-6}{2} = \overset{m}{-3}$

$y - 3 = -3(x-1)$

11. $x^2 + y^2 = 9$ at $(3, 4)$

$2x + 2y\,dy/dx = 0$

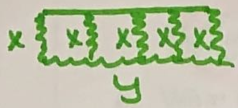
$dy/dx = \frac{-2x}{2y} = \frac{-x}{y}$

$m = \frac{-3}{4}$

$y - 4 = -\frac{3}{4}(x-3)$

Optimization & Related Rates:

12. Four pens will be built side by side along a wall by using 150 feet of fencing. What dimensions will maximize the area of the pens.



$150 = 5x + y$

$y = -5x + 150$

$A = x(-5x + 150)$

$A = -5x^2 + 150x$

$A' = -10x + 150$

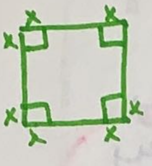
$0 = -10x + 150$

$x = 15'$

$y = -5(15) + 150 = 75'$

$15' \times 75'$

13. A box with an open top is to be constructed from a square piece of cardboard, 3 feet wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.



$H = x$

$L = 3 - 2x$

$W = 3 - 2x$

$V = x(3-2x)(3-2x)$

$V = 9x - 12x^2 + 4x^3$

$V' = 12x^2 - 24x + 9$

$0 = 3(4x^2 - 8x + 3)$

$0 = 3(2x-1)(2x-3)$

$x = 1/2 \quad x = 3/2$

max min

$+ \quad - \quad +$

$1/2 \quad 3/2$

$x = 1/2$

$L + W = 3 - 2(\frac{1}{2}) = 2$

$2' \times 2' \times 1/2'$

14. A travel agency is planning tours for groups of 20 or more. If the group size is exactly 20, the cost is \$400 per person. For each additional person, the price will be reduced by \$15 for every person in the group. What size group will provide the largest revenue?

$R(x)$ = # of people · price/person

$R(x) = (20 + x)(400 - 15x)$

$R(x) = 8000 + 100x - 15x^2$

$R'(x) = -30x + 100$

$0 = -30x + 100$

$x = \frac{10}{3}$

max

$+ + \quad - -$

$10/3$

ppl $= 20 + x$

$= 20 + \frac{10}{3}$

$= 23.3$

23 people

15. The area of a healing wound is given by $A = \pi r^2$. The radius is decreasing at the rate of 1 mm/day. How fast is the area decreasing at the monument when r = 25 mm?

K: $dr/dt = -1$ mm/day

F: dA/dt

W: $r = 25$

$A = \pi r^2$

$dA/dt = 2\pi r \, dr/dt$

$dA/dt = 2\pi(25)(-1)$

$dA/dt = -50\pi$ mm²/day

16. If the radius of a sphere is increasing at the constant rate of 3 mm/sec. How fast is the volume changing when the surface area ($4\pi r^2$) is 10 mm²?

K: $dr/dt = 3$ mm/sec

F: dV/dt

W: SA = 10 mm²

$S = 4\pi r^2$

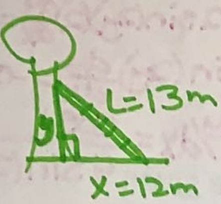
$10 = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 10(3) = 30$ mm³/sec

17. A 13 meter ladder slides down the side of a water tower. At the moment, the bottom end is 12 m from the water tower, the opposite end of the ladder is sliding down at a rate of 3 m/hr. How fast is the bottom of the ladder moving away from the tower?



K: $dy/dt = -3$ m/hr

F: dx/dt

W: $x=12, y=5, L=13$

$x^2+y^2=L^2$

$12^2+y^2=13^2 \quad y=5$m

$x^2+y^2=L^2$

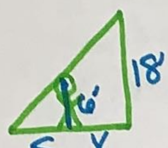
$2x\,dx/dt + 2y\,dy/dt = 2L\,dL/dt$

$2(12)dx/dt + 2(5)(-3) = 2(13)(0)$

$24\,dx/dt = 30$

$dx/dt = 5/4$ m/hr

18. A streetlight is 18 feet above the sidewalk. A man 6 ft. tall walks away from the light at the rate of 15 ft/sec. Determine the rate at which the man's shadow is lengthening at the moment he is 12 ft from the base of the light.



K: $dx/dt = 15$ ft/sec

F: dS/dt

W: $x = 12'$

$\frac{6}{S} = \frac{18}{x+S}$

$18S = 6x + 6S$

$12S = 6x$

$S = \frac{1}{2}x$

$\frac{dS}{dt} = \frac{1}{2}dx/dt$

$\frac{dS}{dt} = \frac{1}{2}(15)$

$\frac{dS}{dt} = \frac{15}{2}$ ft/sec

19. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

K: $dr/dt = 1$ ft/sec

F: dA/dt

W: $r = 4$ft

$A = \pi r^2$

$dA/dt = 2\pi r\,dr/dt$

$dA/dt = 2\pi(4)(1)$

$dA/dt = 8\pi$ ft^2/sec

20. All edges of a cube are expanding at a rate of 6 inches per second. Find the rate of change of the volume of the cube when the area of one side is equal to 25 in^2.

K: $de/dt = 6$ in/sec

F: dV/dt

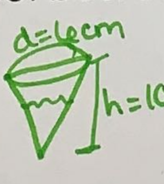
W: $A = 25$in^2 so $e=5$in

$V = e^3$

$dV/dt = 3e^2\,de/dt$

$dV/dt = 3(5)^2(6) = 450$ in^3/sec

21. Suppose you are drinking root beer from a conical paper cup. The cup has a diameter of 6 cm and a depth of 10 cm. As you drink root beer through a straw, the root beer is leaving the cup at a rate of 7 cm^3/sec. At what rate is the level of the root beer in the cup changing when the root beer is 6 cm deep?



K: $dV/dt = -7$ cm^3/sec

F: dh/dt

W: $h = 6$cm $r = 3$cm

$\frac{r}{h} = \frac{3}{10}$

$3h = 10r$

$r = \frac{3}{10}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{3}{10}h\right)^2 h$

$V = \frac{1}{3}\pi \frac{9}{100}h^3$

$V = \frac{3}{100}\pi h^3$

$dV/dt = \frac{9}{100}\pi h^2\,dh/dt$

$-7 = \frac{9}{100}\pi(6)^2(dh/dt)$

$-7 = \frac{81\pi}{25}\,dh/dt$

$\frac{dh}{dt} = \frac{-175}{81\pi}$ cm/sec

Unit 4B Derivative Applications: Additional Review

Use implicit differentiation to find $\frac{dy}{dx}$.

1. $x^2 + y^2 = 16$

$2x + 2y\,dy/dx = 0$

$dy/dx = \frac{-2x}{2y}$

$dy/dx = -x/y$

2. $x^3 - xy + y^2 = x$

$3x^2 - (x\frac{dy}{dx} + y) + 2y\frac{dy}{dx} = 1$

$-x\,dy/dx + 2y\,dy/dx = 1 - 3x^2 + y$

$dy/dx = \frac{-3x^2+y+1}{-x+2y}$

3. $\sin x + 2\cos(2y) = 1$

$\cos x + -2\sin(2y)(2)\frac{dy}{dx} = 0$

$-4\sin(2y)\frac{dy}{dx} = -\cos x$

$dy/dx = \frac{\cos x}{4\sin(2y)}$

Find the equation of the tangent line to the graph of the functions below at the given point.

4. $xy = 4$ @ (-4, -1)

$x\,dy/dx + y = 0$

$dy/dx = \frac{-y}{x} = \frac{1}{-4} = m$

$y + 1 = -\frac{1}{4}(x+4)$

5. $x^3 + y^3 = 2xy$ @ (1,1)

$3x^2 + 3y^2\,dy/dx = 2x\,dy/dx + y \cdot 2$

$3y^2\,dy/dx - 2x\,dy/dx = -3x^2 + 2y$

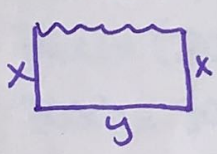
$dy/dx = \frac{-3x^2+2y}{3y^2-2x}$

$m = \frac{-3(1)^2+2(1)}{3(1)^2-2(1)} = \frac{-1}{1}$

$y - 1 = -1(x-1)$

Optimization

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?



$P = 2x + y$

$x = 300m$

$y = \frac{180000}{300}$

$y = 600m$

$A = xy$

$180000 = xy$

$y = \frac{180000}{x}$

300m x 600m

$P = 2x + (\frac{180,000}{x})$

$P = 2x + 180,000x^{-1}$

$P' = 2 - \frac{180,000}{x^2}$

$x^2(0 = 2 - \frac{180,000}{x^2})$

$0 = 2x^2 - 180,000$

$2x^2 = 180,000$

$x^2 = 90000$

$x = 300$

7. A real estate office handles 50 apartment units. When the rent is $720 per month, all units are occupied. However, on the average, for each $40 increase in rent, one unit becomes vacant. Each occupied unit requires an average of $48 per month for services and repairs. What rent should be charged to obtain the maximum profit?

$R(x) = (720 + 40x)(50 - 1x)$

rent · apt #

$R(x) = 36000 + 1280x - 40x^2$

$C(x) = (50 - x)(48)$

apt · cost

$C(x) = 2400 - 48x$

$P(x) = R(x) - C(x) = (36,000 + 1280x - 40x^2) - (2400 - 48x)$

$P(x) = -40x^2 + 1328x + 33,600$

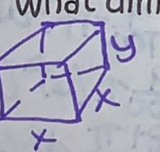
$P'(x) = -80x + 1328$

$0 = -80x + 1328$

$x = 16.6$

Rent = 720 + 40(16.6) = $1,384/month

8. An open rectangular box (no lid) with a square base has volume of 5000 cubic inches. What dimensions yield the minimum surface area? What is the minimum surface area?



$V = x^2y$

$5000 = x^2y$

$y = \frac{5000}{x^2}$

$y = \frac{5000}{(21.5)^2} = 10.8"$

$SA = x^2 + 4xy$

$SA = x^2 + 4x(\frac{5000}{x^2})$

$SA = x^2 + 20,000x^{-1}$

$SA' = 2x - \frac{20,000}{x^2}$

$x^2(0 = 2x - \frac{20000}{x^2})$

$0 = 2x^3 - 20,000$

$2x^3 = 20,000$

$\sqrt[3]{x^3} = \sqrt[3]{10000}$

$x = 21.5"$

$y = 10.8"$

$SA = (21.5)^2 + 4(21.5)(10.8)$

$SA = 1391.05$

$1,391.05\,in^2$

Related Rates

9. A spherical balloon is inflated with gas at the rate of 20 cubic feet per minute. How fast is the radius of the balloon increasing at the instant the radius is 2 feet?

K: $dV/dt = 20\ ft^3/min$ $\quad V = \frac{4}{3}\pi r^3$

F: dr/dt $\quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

W: $r = 2ft$ $\quad 20 = 4\pi(2)^2 dr/dt$

$dr/dt = \frac{20}{16\pi}$ $\quad \frac{dr}{dt} = \frac{5}{4\pi}$ ft/min.

10. At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the height. At what rate is the height of the pile changing when the pile is 15 feet high?

K: $dV/dt = 10ft^3/min$

F: dh/dt

W: $h = 15ft$

$d = 3h$

$2r = 3h$

$r = \frac{3}{2}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{3h}{2}\right)^2 h$

$V = \frac{1}{3}\pi \frac{9}{4}h^2 h$

$V = \frac{3}{4}\pi h^3$

$\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$

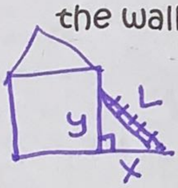
$10 = \frac{9}{4}\pi(15)^2 \frac{dh}{dt}$

$10 = \frac{2025\pi}{4}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{40}{2025\pi}$

$dh/dt = \frac{8}{405\pi}$ ft/min

11. A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second. How fast is the top moving down the wall when the base of the ladder is 7 feet from the wall?



K: $dx/dt = 2ft/sec$

F: dy/dt

W: $x = 7'$, $L = 25'$, $y = 24'$

$x^2 + y^2 = L^2$

$7^2 + y^2 = 25^2$

$\sqrt{y^2} = \sqrt{576} = 24$

$x^2 + y^2 = L^2$

$2x\, dx/dt + 2y\, dy/dt = 2L\, dL/dt$

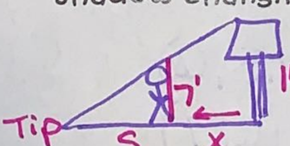
$2(7)(2) + 2(24)dy/dt = 2(25)(0)$

$48 dy/dt = -28$

$dy/dt = -28/48$

$dy/dt = -7/12$ ft/sec.

12. A man 7 feet tall walking at a rate of 7 ft/sec away from a light that is 14 feet above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing?



K: $dx/dt = 7ft/sec$

F: $dTip/dt$ + dS/dt

W:

$\frac{7}{S} = \frac{14}{x+S}$

$14S = 7x + 7S$

$7S = 7x$

$7dS/dt = 7dx/dt$

$7dS/dt = 7(7)$

$dS/dt = 7ft/sec$

$Tip = x + S$

$dTip/dt = dx/dt + dS/dt$

$dTip/dt = 7 + 7$

$\frac{dTip}{dt} = 14$ ft/sec

Particle Motion

Position is measured in meters and time is in seconds for a particle moving along the x-axis.

Given: $s(t) = 2t^3 - 27t^2 + 108t + 5$ [0,7] sec. Find the following:

13. Velocity as a function of time t.

$v(t) = 6t^2 - 54t + 108$

14. Velocity at t = 3 sec.

$v(3) = 6(3)^2 - 54(3) + 108$ $v(3) = 0$ m/s

15. Acceleration as a function of time t.

$a(t) = 12t - 54$

16. When is the particle standing still?

$0 = 6t^2 - 54t + 108$
$0 = 6(t^2 - 9t + 18)$
$0 = 6(t-3)(t-6)$

$t = 3$ sec
$t = 6$ sec

17. When is the particle moving right? Moving left?

+ − + (0, 3, 6, 7)

Right: $(0,3) \cup (6,7)$
Left: $(3,6)$

18. What is the displacement?

$s(0) = 5$ $s(7) = 124$

$124 - 5 = 119$m

19. What is the total distance traveled?

$s(0) = 5$
$s(3) = 140$
$s(6) = 113$
$s(7) = 124$

$(0,3) = 135$
$(3,6) = 27$
$(6,7) = 11$

$135 + 27 + 11 = 173$m

20. When is the particle speeding up? slowing down?

$a(t) = 12t - 54$
$0 = 12t - 54$
$t = 4.5$

$v(t)$: + + | − − − | + + + (0, 3, 6, 7)

$a(t)$: − − − − | + + + + + (0, 4.5, 7)

Slowing down: $(0,3) \cup (4.5,6)$
Speeding up: $(3,4.5) \cup (6,7)$

21. Find the velocity when acceleration is 0.

$a(t) = 12t - 54$
$0 = 12t - 54$
$t = 4.5$

$v(4.5) = 6(4.5)^2 - 54(4.5) + 108$
$= -13.5$ m/s