# The Definition of a Derivative at a Point

## Slope of the Secant Line

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

This formula represents the average rate of change

Review: Find the average rate of change of $f(x) = x^2 - 5x$ from $[2, 5]$.

$f(2) = 4 - 10 = -6$
$f(5) = 25 - 25 = 0$

$$m_{sec} = \frac{0 - -6}{5 - 2} = \frac{6}{3} = 2$$

**SLOPE OF THE TANGENT LINE**

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

This formula represents the **INSTANTANEOUS** rate of change

**DEFINITION OF A DERIVATIVE AT A POINT** $(a, f(a))$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

When to use this version:

-you are given a specific x value to evaluate the derivative at.

**DERIVATIVE**

-Slope of a tangent line at $x = a$
-Rate of change at $x = a$
-Instantaneous rate of change or derivative
-Denoted by $y'$, $f'(x)$, or $\frac{dy}{dx}$

Find the derivative at a given point.

1. $f(x) = 4x + 2$ at $x = -1$

$f(-1) = -2$

$$f'(x) = \lim_{x \to -1} \frac{4x + 2 - (-2)}{x - (-1)}$$

$$\lim_{x \to -1} \frac{4x + 4}{x + 1}$$

$$\lim_{x \to -1} 4$$

$f'(-1) = 4$

2. Find the rate of change at a point.

$f(x) = x^2 - 2$ at $x = 0$ $\quad f(0) = -2$

$$f'(x) = \lim_{x \to 0} \frac{x^2 - 2 + 2}{x - 0}$$

$$\lim_{x \to 0} x$$

$$f'(0) = 0.$$

3. Find the derivative at a given point

$f(x) = \frac{2x+1}{x+2}$ at $(1,1)$

$$f'(x) = \lim_{x \to 1} \frac{\frac{2x+1}{x+2} - 1}{x - 1}$$

$$\lim_{x \to 2} \frac{2x+1-(x+2)}{x+2} \cdot \frac{1}{x-1}$$

$$\lim_{x \to 2} \frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}$$

4. Find the rate of change at a value.

$f(x) = \frac{1}{\sqrt{x}}$ at $x = 9$ $\quad f(9) = 1/3$

$$f'(x) = \lim_{x \to 9} \frac{\frac{1}{\sqrt{x}} - \frac{1}{3}}{x - 9}$$

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{3\sqrt{x}} \cdot \frac{1}{x-9}$$

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{3\sqrt{x}} \cdot \frac{1}{(\sqrt{x}+3)(\sqrt{x}-3)}$$

$$\lim_{x \to 9} \frac{-1}{3\sqrt{x}(\sqrt{x}+3)} = \frac{-1}{3\sqrt{9}(\sqrt{9}+3)} = \frac{-1}{9(6)}$$

$$f'(9) = \frac{-1}{54}$$

# TANGENT AND NORMAL EQUATIONS

To find the **tangent equation**:

-find the slope of the tangent line (find the derivative)

-input the slope and the point into the point-slope form of a line (no need to simplify)

$$y - y_1 = m(x - x_1)$$

To find the equation of the **normal line**:

- Find the slope of the tangent line (find the derivative)
- Find the slope of the line perpendicular to the tangent line (negative reciprocal)
- Input the slope and the point into the point-slope form of a line (no need to simplify)

Find the derivative at the given point. Then find the equation of the tangent + normal line.

1. $f(x) = x^2 + 2x$ when $x = 3$ $\quad f(3) = 15$

$f'(x) = \lim_{x \to 3} \dfrac{x^2 + 2x - 15}{x - 3}$

$\lim_{x \to 3} \dfrac{(x+5)(x-3)}{x-3}$

$\lim_{x \to 3} x + 5 = 3 + 5$

$f'(3) = 8$ $\quad m_{tan} = 8$ $\quad m_{norm} = -\frac{1}{8}$

Tan Line: $y - 15 = 8(x - 3)$

Normal Line: $y - 15 = -\frac{1}{8}(x - 3)$

2. $f(x) = \sqrt{x - 4}$ find $f'(5)$ $\quad f(5) = 1$

$f'(x) = \lim_{x \to 5} \dfrac{\sqrt{x-4} - 1}{x - 5}$

$\lim_{x \to 5} \dfrac{(\sqrt{x-4} - 1)(\sqrt{x-4} + 1)}{(x-5)(\sqrt{x-4} + 1)}$

$\lim_{x \to 5} \dfrac{x - 4 - 1}{(x-5)(\sqrt{x-4} + 1)}$

$\lim_{x \to 5} \dfrac{1}{(\sqrt{x-4} + 1)}$

$f'(5) = 1/2$ $\quad m_{tan} = \frac{1}{2}$ $\quad m_{norm} = -2$

Tan Line: $y - 1 = \frac{1}{2}(x - 5)$

Norm. Line:

$y - 1 = -2(x - 5)$

3. $f(x) = x^3 - 3x + 1$ at $(2, 3)$

$f'(x) = \dfrac{x^3 - 3x + 1 - 3}{x - 2}$

$\lim_{x \to 2} \dfrac{x^3 - 3x - 2}{x - 2}$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -3 & -2 \\ & \downarrow & 2 & 4 & \\ \hline & 1 & 2 & 1 & \end{array}$$

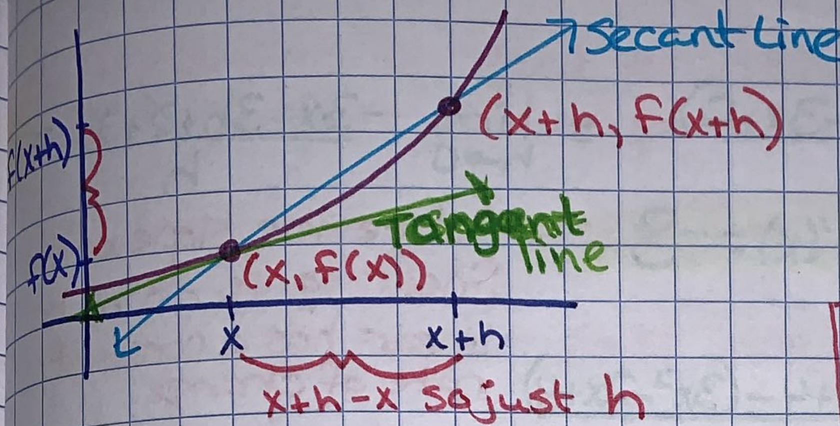
$\lim x^2 + 2x + 1$

$f'(2) = 9$

Tan. Line: $y - 3 = 9(x - 2)$

Norm. Line: $y - 3 = -\frac{1}{9}(x - 2)$

# Definition of a Derivative




$$m_{sec} = \frac{f(x+h) - f(x)}{x+h-x}$$

$$m_{sec} = \frac{f(x+h) - f(x)}{h}$$

$$\boxed{m_{tan} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

Secant line's slope represents average rate of change

Tangent line's slope represents instantaneous rate of change

## Average Rate of Change vs. Instantaneous Rate of $\Delta$

| Similarities | Differences |
|---|---|
| • Both are rates of change | • IROC is a limit; AROC is not |
| • Slopes of lines | • One is an average + one isn't |
| • $\frac{f(x+h) - f(x)}{h}$ | • AROC involves 2 points + IROC involves 1 point |

## Derivative

- slope of a tangent line
- rate of change at one point
- an instantaneous rate of change
- denoted by $y'$, $f'(x)$, or $\frac{dy}{dx}$

**Definition of a derivative:**

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

# Find the derivative using the definition.

1. $f(x) = -3x+2$

$$f'(x) = \lim_{h\to 0} \frac{-3(x+h)+2-(-3x+2)}{h} = \lim_{h\to 0} \frac{-\cancel{3x}-3h+\cancel{2}+\cancel{3x}-\cancel{2}}{h}$$

$$f'(x) = \lim_{h\to 0} -3 \quad \text{so } f'(x) = -3$$

Notice the slope of the linear eq. is -3 ... bc linear has a constant rate of change.

2. $y = 3x^2-2x+4$

$$y' = \lim_{h\to 0} \frac{3(x+h)^2-2(x+h)+4-(3x^2-2x+4)}{h}$$

$$\lim_{h\to 0} \frac{3(x^2+2xh+h^2)-2x-2h+4-3x^2+2x-4}{h}$$

$$\lim_{h\to 0} \frac{\cancel{3x^2}+6xh+3h^2-\cancel{2x}-2h+\cancel{4}-\cancel{3x^2}+\cancel{2x}-\cancel{4}}{h} = \frac{\cancel{h}(6x+3h-2)}{\cancel{h}}$$

$$\lim_{h\to 0} 6x+3(0)-2 \longrightarrow y' = 6x-2$$

3. $y = \dfrac{1}{x+1}$ *Get common denominator

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = \lim_{h\to 0} \frac{\frac{x+1}{(x+h+1)(x+1)} + \frac{\overset{-x-h-1}{-1(x+h+1)}}{(x+h+1)(x+1)}}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\cancel{x}+\cancel{1}-\cancel{x}-\overset{-1}{\cancel{h}}-\cancel{1}}{(x+h+1)(x+1)} \cdot \frac{1}{\cancel{h}} \rightarrow \frac{dy}{dx} = \lim_{h\to 0} \frac{-1}{(x+0+1)(x+1)}$$

$$\frac{dy}{dx} = \frac{-1}{(x+1)^2}$$

4. $f(x) = 3\sqrt{x-2}$ *Rationalize the numerator

$$f'(x) = \lim_{h\to 0} \frac{(3\sqrt{x+h-2} - 3\sqrt{x-2})}{h} \frac{(3\sqrt{x+h-2} + 3\sqrt{x-2})}{(3\sqrt{x+h-2} + 3\sqrt{x-2})}$$

$$= \lim_{h\to 0} \frac{9(x+h-2)-9(x-2)}{h(3\sqrt{x+h-2}+3\sqrt{x-2})}$$

$$= \lim_{h\to 0} \frac{\cancel{9x}+9\cancel{h}-\cancel{18}-\cancel{9x}+\cancel{18}}{\cancel{h}(3\sqrt{x+h-2}+3\sqrt{x-2})} = \lim_{h\to 0} \frac{9}{3\sqrt{x+0-2}+3\sqrt{x-2}}$$

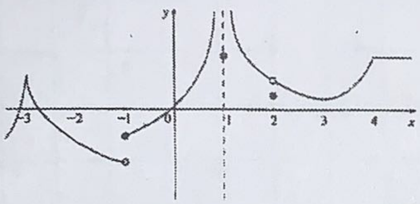
$$f'(x) = \frac{9}{6\sqrt{x-2}} \qquad f'(x) = \frac{3}{2\sqrt{x-2}}$$

# Curve Sketching

A graph is differentiable anywhere except where the graph has the following:

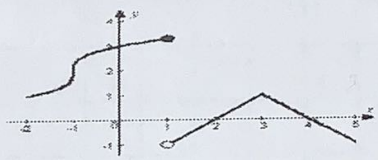
- cusp
- corner
- vertical tangent
- discontinuity (removable, infinite, jump)

STATE THE X VALUES WHERE $f$ IS NOT DIFFERENTIABLE AND THE REASON



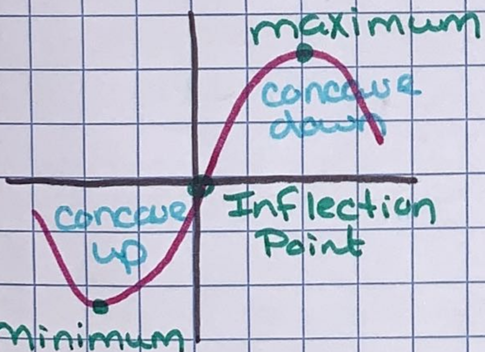

$x=-3$ cusp $x=-1$ jump
$x=1$ infinite $x=2$ hole
$x=4$ corner

STATE THE X VALUES WHERE $f$ IS NOT DIFFERENTIABLE AND THE REASON



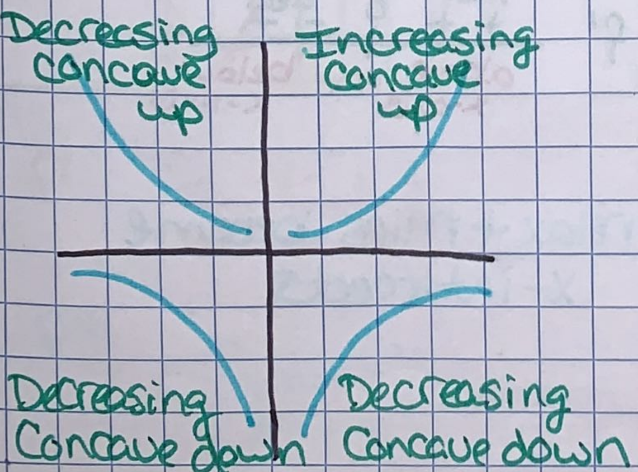
$x=-1$ vertical tangent
$x=3$ Jump discontinuity
$x=3$ corner

## Critical Points, Concavity, + Inflection Points




Critical Points - the graph's turning pts or the local max (peak's) + local min (valley)

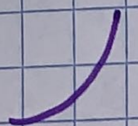
Inflections Points - a pt. on a graph where it changes concavity (1 side is concave up + 1 side is concave down)



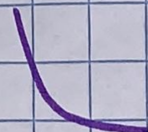

### RELATIONSHIP BETWEEN $f, f', f''$

| $f$ | $f'$ | $f''$ |
|---|---|---|
| -Cusp<br>-Corner<br>-Discontinuity<br>-Removable<br>-Infinite<br>-jump<br>-Vertical Tangent | DNE | DNE |
| Local max, local min (local extrema), horizontal tangent | 0<br>On the x-axis | |
| $f$ increasing | Positive (Above the x-axis) | |
| $f$ decreasing | Negative (Below the x-axis) | |
| $f$ concave up | Increasing | Positive (Above the x-axis) |
| $f$ concave down | Decreasing | Negative (Below the x-axis) |
| Points of Inflections | Local Extrema | Change Signs |

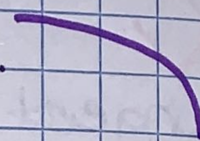
What can we say about g, g', g" for each segment of the graph?

1. 

g: Incr / Conc Up
g': + / Incr
g": + (above x-axis)

2. 

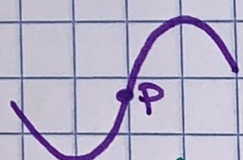
g: Decr / Conc Up
g': − / Incr
g": +

3. 

g: Decr / Conc ↓
g': − / Decr
g": − (below) x-axis

4. 

g: Incr / Conc down
g': + / Decr
g": −

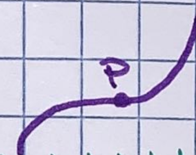
5. 


g' − − o + + o − −

g" + + + o − − −

bc conc up   conc down

+ means above x-axis
− means below x-axis

6. 

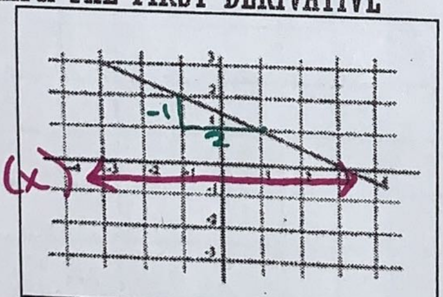

g' + + + + + + + Incr everywhere

g" − − o + + +

bc conc ↓   conc up

★ concavity tells you whether the 2nd derivative of a function is positive (above x-axis) or negative (below)

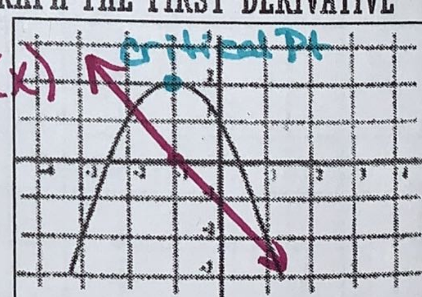
1. GRAPH THE FIRST DERIVATIVE




f' decreasing − − − −   all below x-axis

m = $-\frac{1}{2}$

2. GRAPH THE FIRST DERIVATIVE

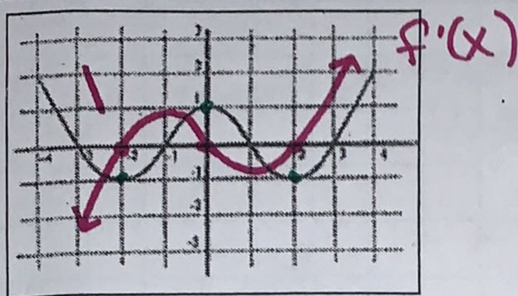



f' incr + + o decr − −

above x-axis   on   below x-axis

★ Max + mins become x-intercepts

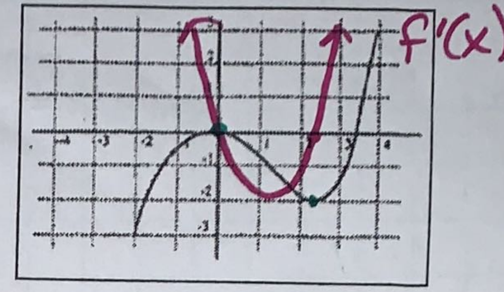
## 3. GRAPH THE FIRST DERIVATIVE




f' − 0 + 0 − 0 +

below −2 ab 0 bel 2 above

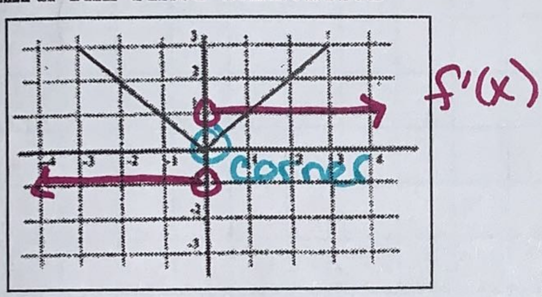
## 4. GRAPH THE FIRST DERIVATIVE




f' + 0 − 0 +

above 0 below 2.1 above

## 5. GRAPH THE FIRST DERIVATIVE



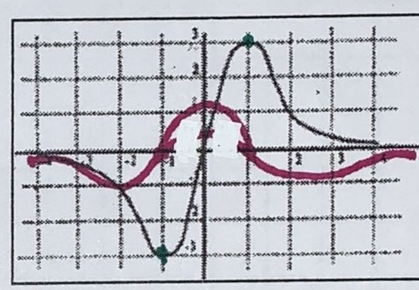

f' decr − − undef incr + +

m = −1 0 m = 1

Think 2 linear graphs

2 constant slopes

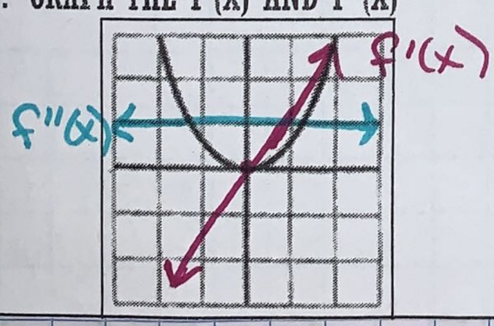
## 6. GRAPH THE FIRST DERIVATIVE



f' close to 0 − 0 + 0 − close to 0

close to 0 below −1 above 1 below close to 0

## 7. GRAPH THE F'(X) AND F''(X)




f' − − 0 + +

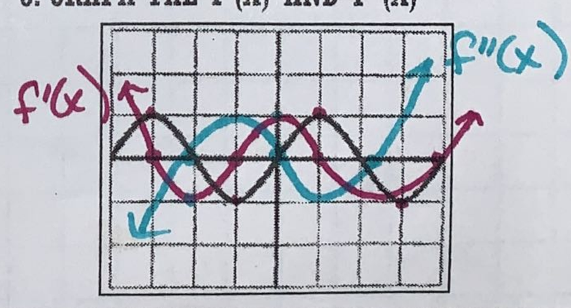
below 0 above

f'' − − −

below

m = 1

## 8. GRAPH THE F'(X) AND F''(X)




f' + 0 − 0 + 0 − 0 +

ab −3 bel −1 1 3

f'' − 0 + 0 − 0 +

−2 0 2.2

44 Curve Sketching f from f'

What does the 1st derivative of f(x) tell you about f(x)?

| f'(x) | + | − | 0 | f(x) defined but f'(x) undefined |
|---|---|---|---|---|
| f(x) | Increase | Decrease | Critical Pt. or Horiz. Tan | |

If "a" is a critical value, what can we tell about f(x)?

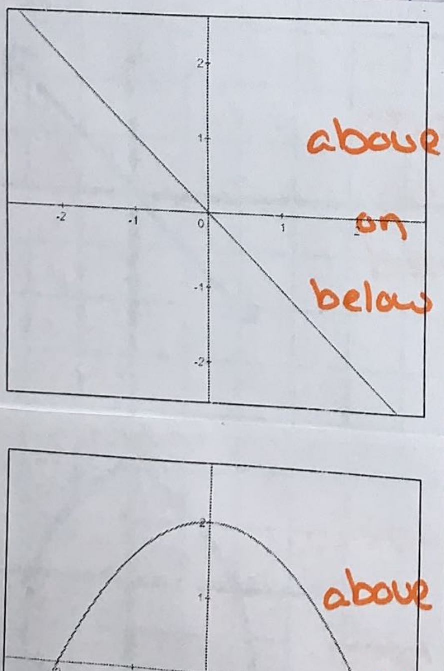
| f'(x) | + 0 or − ; a | − 0 + ; a |
|---|---|---|
| f(x) | local max or undef. | local min or undef. |

What does the 2nd derivative tell us about f(x)?

| f''(x) | + | − | 0 or undefined |
|---|---|---|---|
| f(x) | concave up | concave down | possible pt. of inflection |

From f', sketch f and f''.

1.



above

on

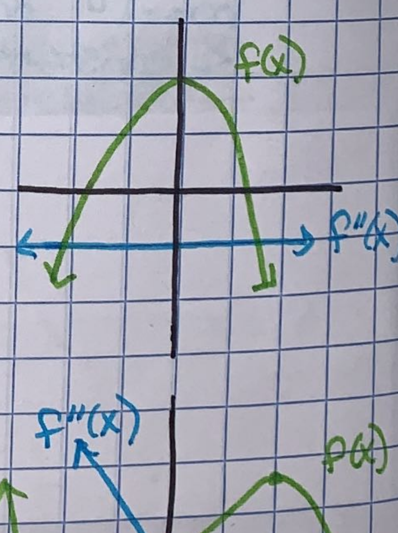
below

② f max

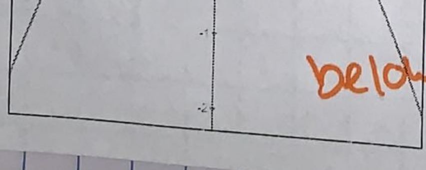
① f' + + 0 − − ; a

③ f'' − − − − ; m = −1

look at f' graph




2.



above

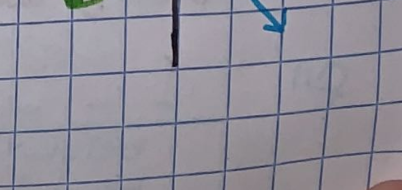
on

below

② f min max

① f' − 0 + + 0 − ; −2 2

③ f'' + + 0 − − ; 0



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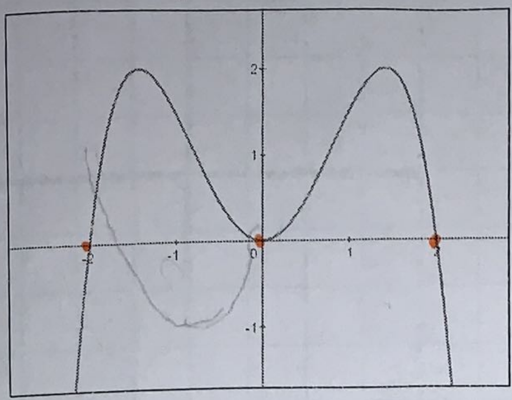
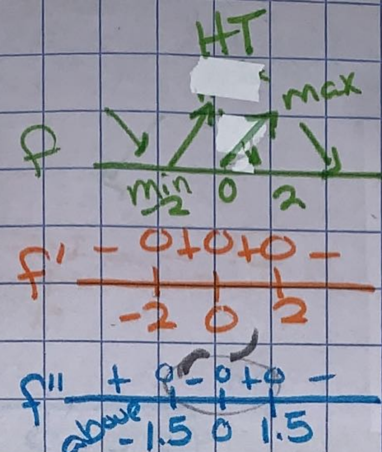
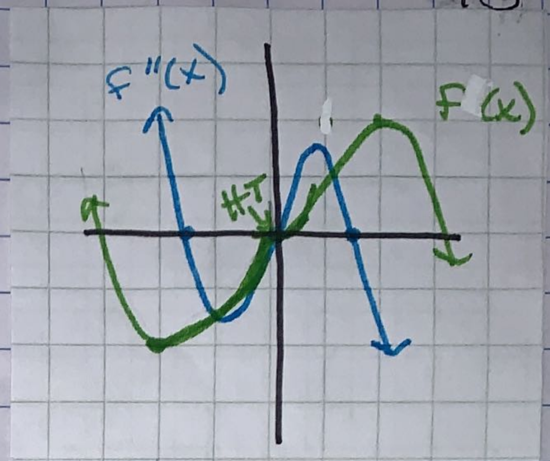
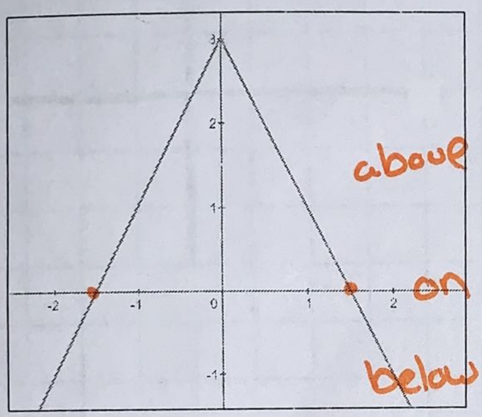
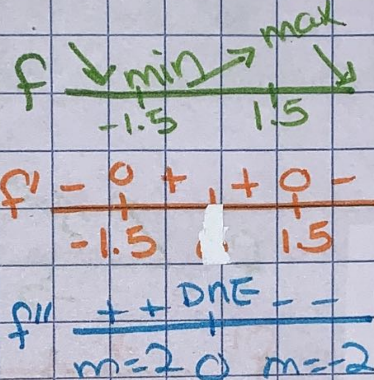


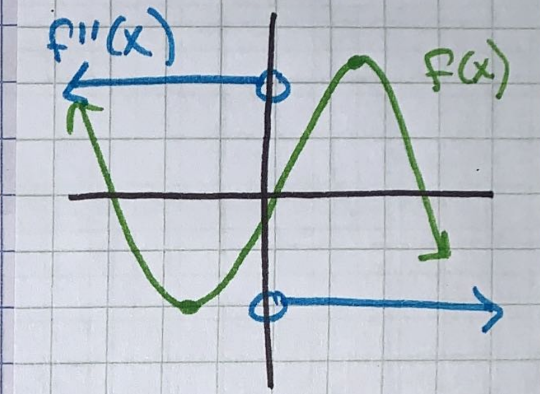



4.

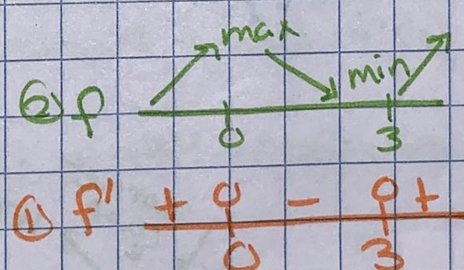


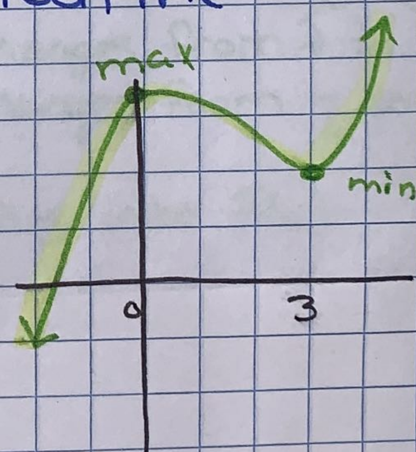



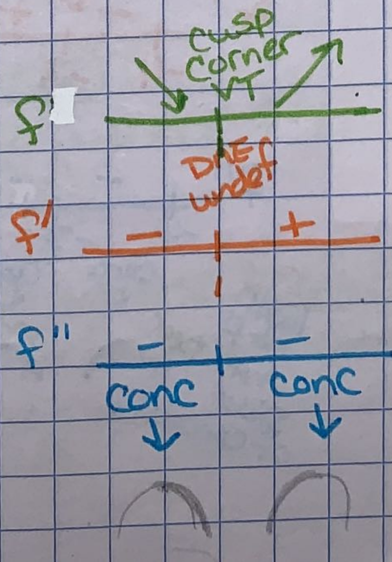

Draw a possible graph of f(x) given the following:

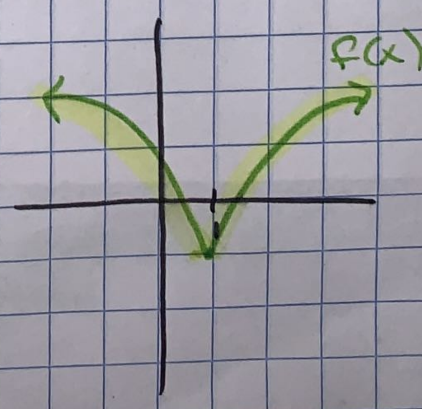
1. a. $f(x)$ is continuous
b. $f(3)=2$ (3,2)
c. $f'(x)>0$, $(-\infty,0)$,$(3,\infty)$
d. $f'(x)<0$, $(0,3)$
e. $f'(x)=0$ at $x=0$ ,$x=3$



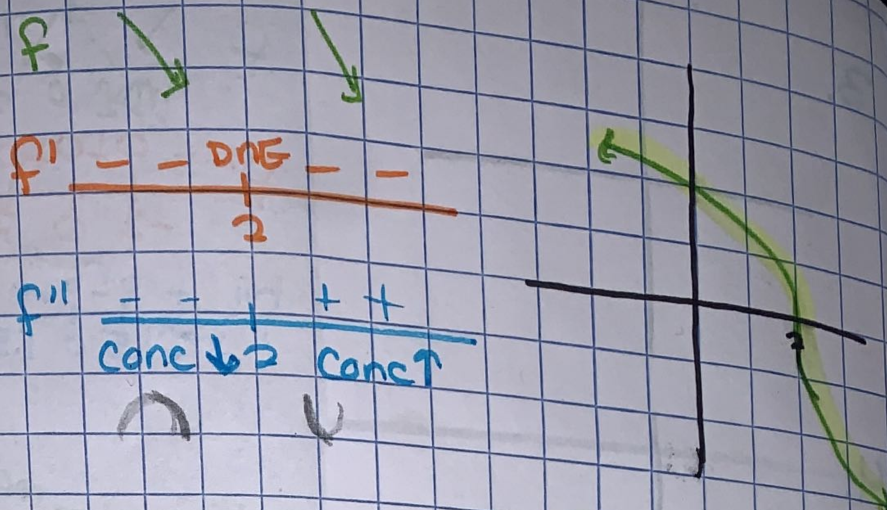




2. a. $f(x)$ is continuous
b. $f'(x)<0$, $(-\infty,1)$
c. $f'(x)>0$, $(1,\infty)$
d. $f'(x)=$ undef. at $x=1$
e. $f''(x)<0$ at $(-\infty,1)\cup(1,\infty)$

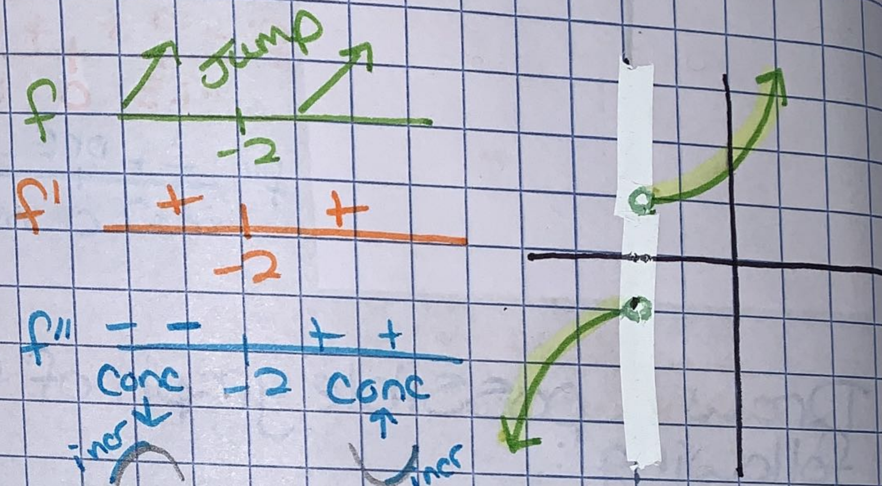


**3.**
a. $f(x)$ is continuous
b. $f'(x)<0; (-\infty,2),(2,\infty)$
c. $f'(x)$ is undefined at $x=2$
d. $f''(x)<0$ when $x<2$
e. $f''(x)>0$ when $x>2$




**4.**
a. $f(x)$ has jump discont. at $x=-2$
b. $f'(x)>0; (-\infty,-2),\cup(-2,\infty)$
c. $f''(x)<0; (-\infty,-2)$
d. $f''(x)>0; (-2,\infty)$




**5.**
a. $f(x)$ is continuous $[-4, 3]$
b. $f'(x)<0$ on $(-4,-2)$
c. $f'(x)>0$ on $(-2,1)\text{U}(1,3)$
d. $f'(x)=$ undef. at $x=-2$
e. $f(-2)=-3$ $f(1)=3$
f. $f'(x)=0$ at $x=1$
g. $f''<0$ on $(-4,-2)\text{U}(-2,1)$
h. $f''>0$ on $(1,3)$