18 Finding Limits from Graphs

DEFINITION OF A LIMIT

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the **limit** of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x \to c} f(x) = L$$

LIMITS FROM A TABLE

From left ⟶ x ⟵ From Right

x	8.9	8.99	8.999	8.9999	9	9.001	9.01	9.1
f(x)	5.98329	5.99883	5.99983	5.999983	6	6.00016	6.00166	6.016
g(x)	15.21	15.9201	15.99201	15.999200	und	16.00080	16.0801	16.81
h(x)	5.98329	5.99883	5.99983	5.999983	6	16.00080	16.0801	16.81

Find the following limits:

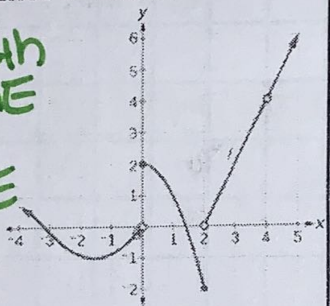
(a) $\lim_{x \to 9} f(x)$ 6 (b) $\lim_{x \to 9} g(x)$ 16 (c) $\lim_{x \to 9} h(x)$ DNE

EX 1: EVALUATING LIMITS FROM A GRAPH

a. $\lim_{x \to 0^-} f(x)$ left 0
b. $\lim_{x \to 0^+} f(x)$ right 2
c. $\lim_{x \to 0} f(x)$ both DNE

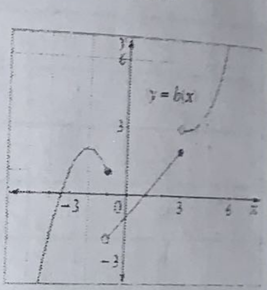
d. $\lim_{x \to 2^-} f(x)$ -2
e. $\lim_{x \to 2^+} f(x)$ 0
f. $\lim_{x \to 2} f(x)$ DNE

g. $\lim_{x \to 4^-} f(x)$ 4
h. $\lim_{x \to 4^+} f(x)$ 4
i. $\lim_{x \to 4} f(x)$ 4



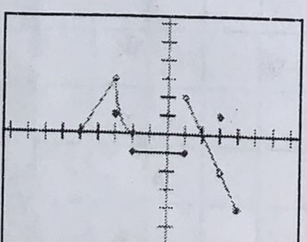
EX 2: EVALUATING LIMITS FROM A GRAPH

a. $\lim_{x \to -1^-} h(x)$ 1
b. $\lim_{x \to -1^+} h(x)$ -2
c. $\lim_{x \to -1} h(x)$ DNE
d. $h(-1)$ 1
e. $\lim_{x \to 3^-} h(x)$ 2
f. $\lim_{x \to 3^+} h(x)$ 3
g. $\lim_{x \to 3} h(x)$ DNE
h. $h(3)$ 2



EX 3: EVALUATING LIMITS FROM A GRAPH

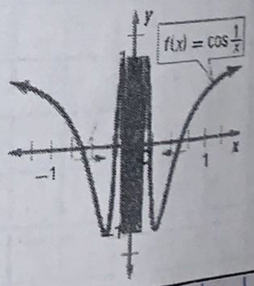
1. $\lim_{x \to 3} g(x)$ -2
2. $\lim_{x \to 0} g(x)$ -1
3. $\lim_{x \to -3} g(x)$ 3
4. $\lim_{x \to 1^+} g(x)$ 2
5. $\lim_{x \to 1^-} g(x)$ -1
6. $\lim_{x \to 1} g(x)$ DNE
7. $\lim_{x \to -2^+} g(x)$ -1
8. $\lim_{x \to 4} g(x)$ DNE
9. $\lim_{x \to 2} g(x)$ 0
10. $\lim_{x \to -2^-} g(x)$ 0



EXAMPLE WITH OSCILLATION

$$\lim_{x \to 0} \cos \frac{1}{x}$$

DNE



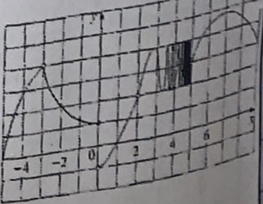
CONDITIONS UNDER WHICH LIMITS DO NOT EXIST

The limit of $f(x)$ as $x \to c$ does not exist under any of the following conditions:

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c.
2. $f(x)$ increases and decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

EX 4: EVALUATE THE LIMIT

a. $\lim_{x \to -3^-} h(x)$ 4
b. $\lim_{x \to -3^+} h(x)$ 4
c. $\lim_{x \to -3} h(x)$ 4
d. $h(-3)$ DNE
e. $\lim_{x \to 0^-} h(x)$ 1
f. $\lim_{x \to 0^+} h(x)$ -1
g. $\lim_{x \to 0} h(x)$ DNE
h. $h(0)$ 1
i. $\lim_{x \to 2} h(x)$ 2
j. $h(2)$ DNE
k. $\lim_{x \to 5^+} h(x)$ 3
l. $\lim_{x \to 5^-} h(x)$ DNE



Graphs From Limits

- Draw in any pt value $f(\;) = \#$
- Draw in any asymptotes
  - Vertical : $\lim_{x \to \#} f(x) = \pm\infty$ - Horizontal : $\lim_{x \to \pm\infty} f(x) = \#$
- Draw arrows
- Connect

★ Be sure that the graph drawn is a function! It must pass the vertical line test.

EXAMPLE 1 – CREATE THE GRAPH GIVEN THE LIMITS

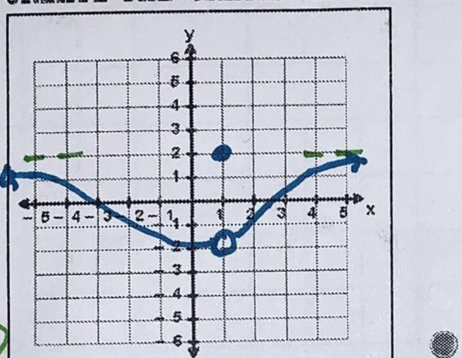
Given:

$\lim_{x \to \infty} f(x) = 2$ HA

$\lim_{x \to 1} f(x) = -2$ both

$\lim_{x \to -\infty} f(x) = 2$ HA

$f(1) = 2$ pt (1,2)



EXAMPLE 2 – CREATE THE GRAPH GIVEN THE LIMITS

Given:

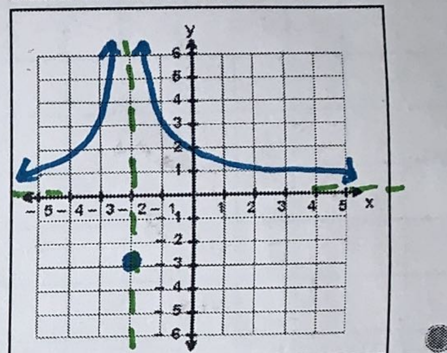
$\lim_{x \to \infty} f(x) = 0$ HA

$\lim_{x \to -2^+} f(x) = \infty$ VA

$\lim_{x \to -2^-} f(x) = \infty$ VA

$\lim_{x \to -\infty} f(x) = 0$ HA

$f(-2) = -3$ (-2,-3)



EXAMPLE 3 – CREATE THE GRAPH GIVEN THE LIMITS

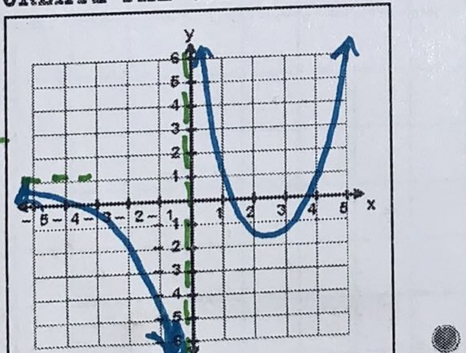
Given:

$\lim_{x \to \infty} f(x) = \infty$ End Behavior

$\lim_{x \to 0^+} f(x) = \infty$ VA

$\lim_{x \to 0^-} f(x) = -\infty$ VA

$\lim_{x \to -\infty} f(x) = 1$ HA



EXAMPLE 4 – CREATE THE GRAPH GIVEN THE LIMITS

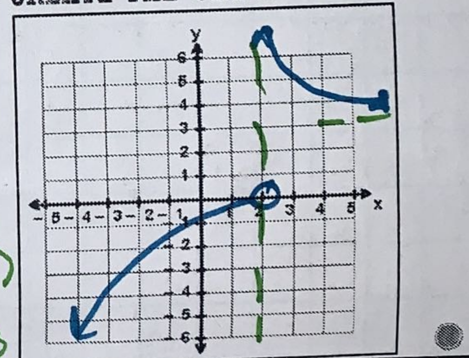
Given:

$\lim_{x \to \infty} f(x) = 3$ HA

$\lim_{x \to 2^+} f(x) = \infty$ VA

$\lim_{x \to 2^-} f(x) = 0$ (2,0) from left

$\lim_{x \to -\infty} f(x) = -\infty$ EB



EXAMPLE 5 – CREATE THE GRAPH GIVEN THE LIMITS

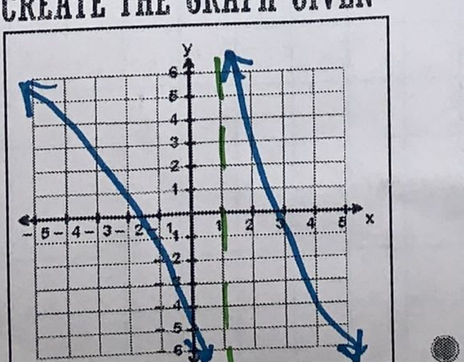
Given:

$\lim_{x \to \infty} f(x) = -\infty$ EB

$\lim_{x \to 1^+} f(x) = \infty$ VA

$\lim_{x \to 1^-} f(x) = -\infty$ VA

$\lim_{x \to -\infty} f(x) = \infty$ EB



EXAMPLE 6 – CREATE THE GRAPH GIVEN THE LIMITS

Given:

$\lim_{x \to \infty} f(x) = 2$ HA

$\lim_{x \to -2} f(x) = -4$ both

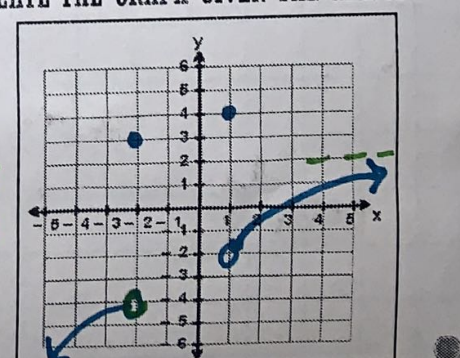
$\lim_{x \to 1^+} f(x) = -2$ right

$\lim_{x \to 1^-} f(x) = -\infty$ VA left

$\lim_{x \to -\infty} f(x) = -\infty$ EB

$f(-2) = 3$ pt.

$f(1) = 4$ pt.



20 # One-Sided Limits

$x \to a^+$ means x is approaching from the right
$x \to a^-$ means x is approaching from the left

Find the limit.

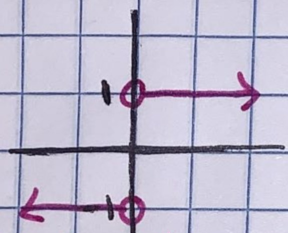
1. $\lim_{x \to 5^+} \frac{x-5}{x^2-25}$ ★ If you plug it in, it DNE

$$\frac{\cancel{x-5}}{(x+5)\cancel{(x-5)}} = \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$$

2. $\lim_{x \to 0^-} \frac{|x|}{x} = -1$

x	y
-2	-1
-1	-1
0	DNE

or you can simplify the problem + use the sign of the limit



3. $\lim_{x \to 0^+} \frac{|x|}{x} = 1$

x	y
2	1
1	1
0	DNE

4. $\lim_{x \to 3^-} \frac{1}{x-3} = -\infty$

x	y
2	-1
2.9	-10
2.99	-100
2.999	-1000

or you can just pick a # really close to the x value + plug in since we know it's a V.A. + it will be ∞ or $-\infty$

5. $\lim_{x \to -3^+} \frac{-5x^2-1}{x^2-9} = \infty$

$$\frac{-5(-2.999)^2-1}{(-2.999)^2-9} = 7662.9$$ which is really big so ∞

6. $f(x) = \begin{cases} x^2+1, & x<0 \\ -2x+4, & 0 \le x < 2 \\ (x-2)^2+1, & x>2 \end{cases}$

a) $\lim_{x \to 0^-} (0)^2+1 = 1$ ← $x<0$

b) $\lim_{x \to 0}$ $x \to 0^- = 1$ $x \to 0^+ = 4$ so DNE
← both so plug into $x<0$ + $x>0$

c) $\lim_{x \to 2^+} (2-2)^2+1 = 1$ $x>2$

d) $\lim_{x \to 2^-} -2(2)+4 = 4$

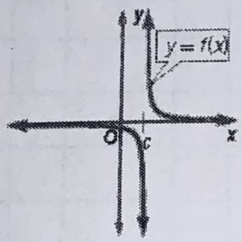
e) $\lim_{x \to 1}$ $\lim_{x \to 1^-} = 2$ $\lim_{x \to 1^+} = 2$ so 2

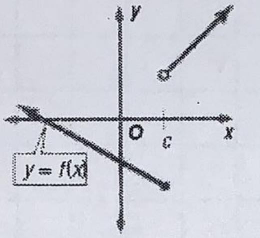
Continuity

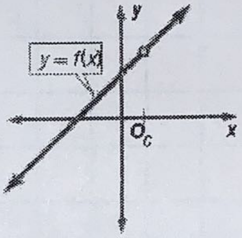
The graph of continuous functions has no breaks, holes or gaps. You can trace the graph of a continuous function without lifting your finger.

Types of Discontinuity:

A function has an **infinite discontinuity** at $x = c$ if the function value increases or decreases indefinitely as x approaches c from the left and right.	A function has a **jump discontinuity** at $x = c$ if the limits of the function as x approaches c from the left and right exist but have two distinct values.	A function has a **removable discontinuity** if the function is continuous everywhere except for a hole at $x = c$.
Example	**Example**	**Example**





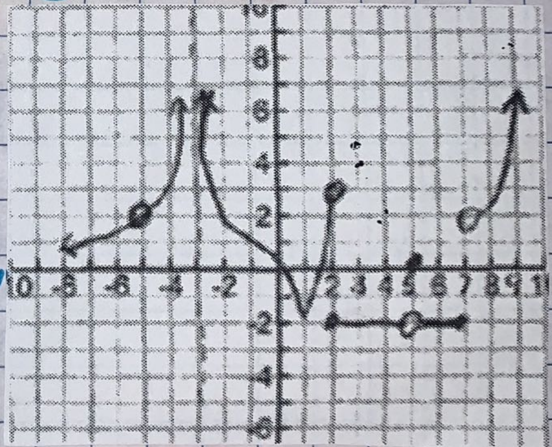


Continuity Test:

A function is continuous at $x = c$ if it satisfies the following...

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c} f(x) = f(c)$$

Using the graph, answer the following:



a) Does $f(5)$ exist? yes; 0

b) Does $\lim_{x \to 5} f(x)$ exist? yes; -2

c) Is $f(x)$ continuous at $x = 5$? Justify.
no; $f(5) \neq \lim_{x \to 5} f(x)$

d) What new value should be assigned to $f(5)$ to remove the discontinuity? -2

e) Does $f(2)$ exist? yes; at -2

f) Does $\lim_{x \to 2} f(x)$ exist? no

g) Does $f(-5)$ exist? no

h) Does $\lim_{x \to -5} f(x)$ exist? yes; 2

i) Is $f(x)$ continuous at $x = -5$? Justify. no; $f(-5) \neq \lim_{x \to -5} f(x)$

j) What new value should be assigned to $f(-5)$ to remove the discontinuity? 2

k) Is $f(x)$ right continuous, left continuous, or neither at $x = 2$? right bc $\lim_{x \to 2^+} f(x) = f(2)$ $x = 7$? left cont. bc $\lim_{x \to 7^-} f(x) = -2 = f(7)$

l) List all the places where $f(x)$ is discontinuous + state the type

$x = -5$ Removable
$x = -3$ Infinite
$x = 2$ Jump
$x = 5$ Removable
$x = 7$ Jump

22

Identify the type of discontinuity

a) $h(x) = \frac{6}{x-3}$

Infinite (VA @ $x=3$)

b) $p(x) = \begin{cases} 3x-1, & x \geq 1 \\ 4x-2, & x < 1 \end{cases}$

$\lim_{x\to 1^-} = 2$ $\lim_{x\to 1^+} = 2$ $\lim_{x\to 1} = 2$

$f(1) = 2$ Continuous

c) $m(x) = \begin{cases} 2x-5, & x \geq 2 \\ 3x, & x < 2 \end{cases}$

$\lim_{x\to 2^-} = -6$ $\lim_{x\to 2^+} = -1$

Jump at $x=2$

d) $h(x) = \frac{6x-2}{9x-3}$

$\frac{2\cancel{(3x-1)}}{3\cancel{(3x-1)}}$ hole in graph at 1/3

Removable at $x = 1/3$

e) $j(x) = \frac{2x-4}{x^2-2x}$ $\frac{2\cancel{(x-2)}}{x\cancel{(x-2)}}$ hole @ $x=2$ VA @ $x=0$

Removable at $x=2$ + Infinite at $x=0$

Finding Values for Discontinuities

1. Find a value for 'a' so that f(x) is continuous

$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ ax+1, & \text{if } x > 2 \end{cases}$ *plug in 2 for x + set = to each other

$2x+3 = ax+1$

$2(2)+3 = a(2)+1$

$7 = 2a+1$

$a = 3$

Find the Intervals on which the Function is Continuous

1. $f(x) = \frac{x-3}{x^2-9} = \frac{\cancel{x-3}}{(x+3)\cancel{(x-3)}} = \frac{1}{x+3}$ hole at $x=3$ VA at $x=-3$ $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

2. $f(x) = \begin{cases} x^2, & x \geq 0 \\ -3, & x < 0 \end{cases}$ $\lim_{x\to 0^+} = 0$ $\lim_{x\to 0^-} f(x) = -3$ $(-\infty,0)\cup(0,\infty)$

3. $f(x) = x^2-x-12$ $(-\infty,\infty)$

Algebraic Limits (2.5)

Limits Strategy Flowchart

Try to evaluate $f(a)$. What did you get?

Try direct substitution 1st

I got a number, c #

done!

You're done! $\lim_{x \to a} f(x) = c$

I got $\frac{\text{not } 0}{0}$ $= \frac{\#}{0}$ so

The graph has a vertical asymptote of $x = a$. $\lim_{x \to a} f(x)$ is not a finite number. Your answer is $+\infty$, $-\infty$, or DNE. Check left-hand and right-hand limits to determine which it is.

∞, $-\infty$, or DNE

I got $\frac{0}{0}$ $\frac{0}{0}$ or $\frac{\infty}{\infty}$

"play time"

This is an indeterminate form, the signal that you need to *do something* to change the form of the expression.
Here are some options:

- *Factor and reduce, if appropriate*
- *Rationalize the numerator or denominator, if appropriate*
- *Apply L'Hospital's Rule*
- *Apply trig identities*

When you no longer have an indeterminate form, return to the top and start again with your new expression.

1. $\lim_{x \to 4} (x^2 - 6x + 3)$

$(4)^2 - 6(4) + 3 = -5$

2. $\lim_{x \to 2} \frac{4x^3 + 1}{x - 5}$

$\frac{4(2)^3 + 1}{2 - 5} = \frac{33}{-3} = -11$

3. $\lim_{x \to 2} \frac{x - 3}{2x^2 - x - 15}$

$\frac{2 - 3}{2(2)^2 - 2 - 15} = \frac{-1}{-9} = \frac{1}{9}$

4. $\lim_{x \to -4} \frac{x^2 - x - 20}{x + 4} = \frac{0}{0}$ Factor

$\frac{(x-5)\cancel{(x+4)}}{\cancel{x+4}} = x - 5$

$-4 - 5$

-9

Cont. Algebraic Limits . . .

5. $\lim\limits_{x\to 3} \dfrac{x-3}{x^3-3x^2-7x+21} = \dfrac{0}{0}$

$\dfrac{x-3}{x^2(x-3)-7(x-3)}$

$\dfrac{\cancel{x-3}}{(x^2-7)\cancel{(x-3)}} = \dfrac{1}{x^2-7} = \dfrac{1}{2}$

3^2-7

6. $\lim\limits_{x\to 6} \dfrac{x^2-7x+6}{3x^2-11x-42} = \dfrac{0}{0}$

$\dfrac{\cancel{(x-6)}(x-1)}{(3x+7)\cancel{(x-6)}}$

$\dfrac{6-1}{3\cdot 6+7} = \dfrac{5}{25} = \dfrac{1}{5}$

7. $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9} = \dfrac{0}{0}$ $\boxed{\text{Creative factoring}}$

$\dfrac{\cancel{\sqrt{x}-3}}{(\sqrt{x}+3)\cancel{(\sqrt{x}-3)}} = \dfrac{1}{\sqrt{x}+3}$

$\dfrac{1}{\sqrt{9}+3} = \dfrac{1}{6}$

★Rationalize numerator

8. $\lim\limits_{x\to 0} \dfrac{(2-\sqrt{x+4})}{x}\cdot\dfrac{(2+\sqrt{x+4})}{(2+\sqrt{x+4})}$

$\dfrac{4+\cancel{2\sqrt{x+4}}-\cancel{2\sqrt{x+4}}-(x+4)}{x(2+\sqrt{x+4})}$

$\dfrac{\cancel{4}-1\cancel{x}-\cancel{4}}{\cancel{x}(2+\sqrt{x+4})} = \dfrac{-1}{2+\sqrt{x+4}}$

$\dfrac{-1}{2+\sqrt{0+4}} = -\dfrac{1}{4}$

★Clear "little denominators"

9. $\lim\limits_{x\to 3} \dfrac{\frac{3\cdot 1}{3\cdot x} - \frac{1\cdot x}{3\cdot x}}{x-3} = \dfrac{0}{0}$

$\dfrac{\frac{3}{3x} - \frac{x}{3x}}{x-3} = \dfrac{\frac{3-x}{3x}}{x-3}$

$\dfrac{3-x}{3x} \div \dfrac{x-3}{1} = \dfrac{-1\cancel{(x-3)}}{3x}\cdot\dfrac{1}{\cancel{x-3}}$

$\dfrac{-1}{3x} = \dfrac{-1}{3\cdot 3} = \dfrac{-1}{9}$

10. $\lim\limits_{x\to 0} \dfrac{\frac{2\cdot 1}{2\cdot(2+x)} - \frac{1\cdot(x+2)}{2\cdot(x+2)}}{x}$

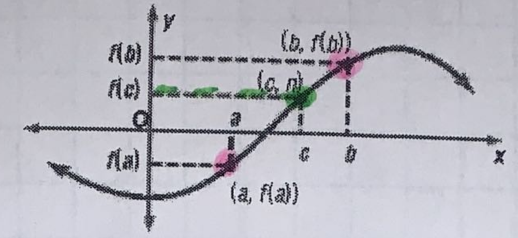
$\dfrac{\frac{2}{2(x+2)} + \frac{-x-2}{2(x+2)}}{x} = \dfrac{\frac{-x}{2(x+2)}}{x}$

$\dfrac{-x}{2(x+2)} \div \dfrac{x}{1} = \dfrac{-\cancel{x}}{2(x+2)}\cdot\dfrac{1}{\cancel{x}}$

$\dfrac{-1}{2(x+2)} = \dfrac{-1}{2(0+2)} = \dfrac{-1}{4}$

Intermediate Value Theorem

If $f(x)$ is a continuous function and $a < b$ and there is a value n such that n is between $f(a)$ and $f(b)$, then there is a number c, such that $a < c < b$ and $f(c) = n$.



Corollary: The Location Principle If $f(x)$ is a continuous function and $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c, such that $a < c < b$ and $f(c) = 0$. That is, there is a zero between a and b.

1. Determine between which consecutive integers the real zeros are located within the interval $[-4, 4]$ of $f(x) = x^3 - 4x + 2$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-46	-13	2	5	2	-1	2	17	50

\- +   + - +

Between
-3 + -2
0 + 1
1 + 2

2. Verify that the IVT applies to the indicated interval & find the value of c guaranteed by the theorem. $f(x) = x^2 + x - 1$ $[0, 5]$ $f(c) = 11$

$f(0) = -1$ & $f(5) = 29$ so $-1 < 11 < 29$ ✓

Find value of c: $11 = x^2 + x - 1$
$0 = x^2 + x - 12$
$0 = (x+4)(x-3)$
$x \neq -4$ $x = 3$

It can't be -4 bc it's not in the interval $[0,5]$

3. Verify that the IVT applies & find c.
$f(x) = \frac{x^2 + x}{x - 1}$ $[5/2, 4]$ $f(c) = 6$

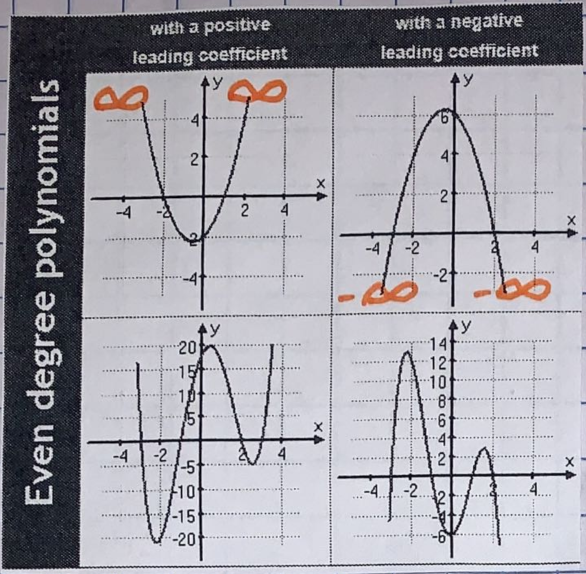
$f(5/2) = \frac{\frac{25}{4} + \frac{5}{2}}{\frac{5}{2} - 1} = \frac{35}{4} \cdot \frac{2}{3} = \frac{35}{6} \approx 5.8\overline{3}$

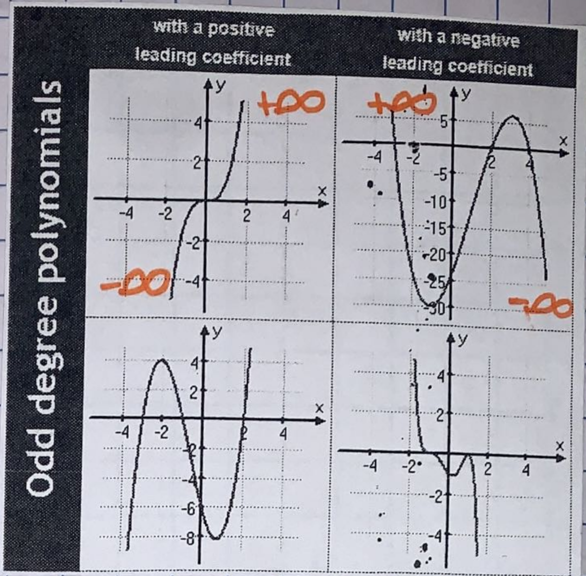
$f(4) = \frac{16+4}{4-1} = \frac{20}{3} \approx 6.\overline{6}$

$5.8\overline{3} < 6 < 6.\overline{6}$ ✓

$\frac{6}{1} = \frac{x^2+x}{x-1}$
$x^2 + x = 6x - 6$
$x^2 - 5x + 6 = 0$
$(x-3)(x-2) = 0$
$x = 3$ $x \neq 2$

End Behavior & Asymptotes





Describe the end behavior:

1. $-x^7+2x^5-4x^2+2x-4$ → odd degree, negative L.C.
   $x \to +\infty,\ f(x) \to -\infty$ $x \to -\infty,\ f(x) \to \infty$
2. $x^8+2x^5-4x^{10}+2x-4$ → even degree, negative L.C.
   $x \to +\infty,\ f(x) \to -\infty$ $x \to -\infty,\ f(x) \to -\infty$
3. $-x^7+2x^5-4x^2+2x^9-4$ → odd degree, positive L.C.
   $x \to +\infty,\ f(x) \to +\infty$ $x \to -\infty,\ f(x) \to -\infty$

Asymptotes of Rational Functions

Vertical Asymptotes: set denominator = 0 & solve for x

Horizontal Asymptotes:

- degree of numerator = degree of denominator
  HA: y = ratio of leading coefficients
- degree of numerator < degree of denominator
  HA: $y = 0$
- degree of numerator > degree of denominator
  HA: none (but there could be a slant asymptote)

Oblique/Slant Asymptote: only when numerator's degree is 1 higher than denom. deg. ★ use long or synthetic ÷ to find slant asymptotes

★★★ Watch out for holes!

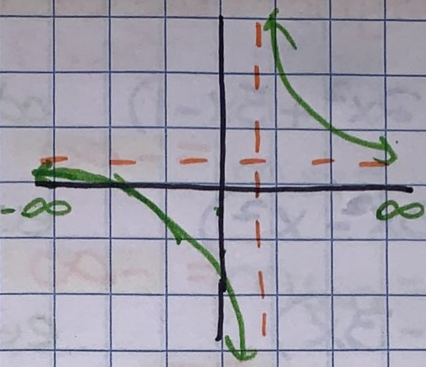
Find all asymptotes. Sketch + discuss end behavior.

1. $f(x) = \frac{4x+5}{8x-3}$

UA: $8x - 3 = 0$

$x = 3/8$

HA: $y = 4/8$ or $1/2$



as $x \to \infty$, $f(x) \to 1/2$

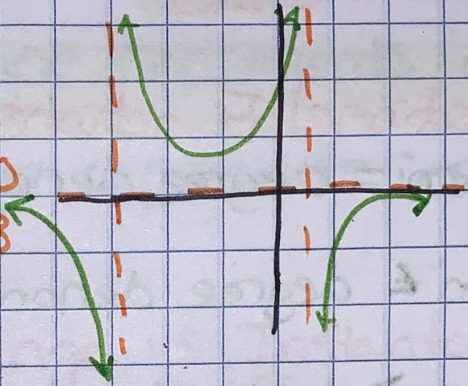
$x \to -\infty$, $f(x) \to 1/2$

2. $f(x) = \frac{x-12}{2x^2+5x-3}$

$(2x-1)(x+3) = 0$

VA: $x = 1/2$ $x = -3$

HA: $y = 0$



as $x \to \infty$, $f(x) \to 0$

$x \to -\infty$, $f(x) \to 0$

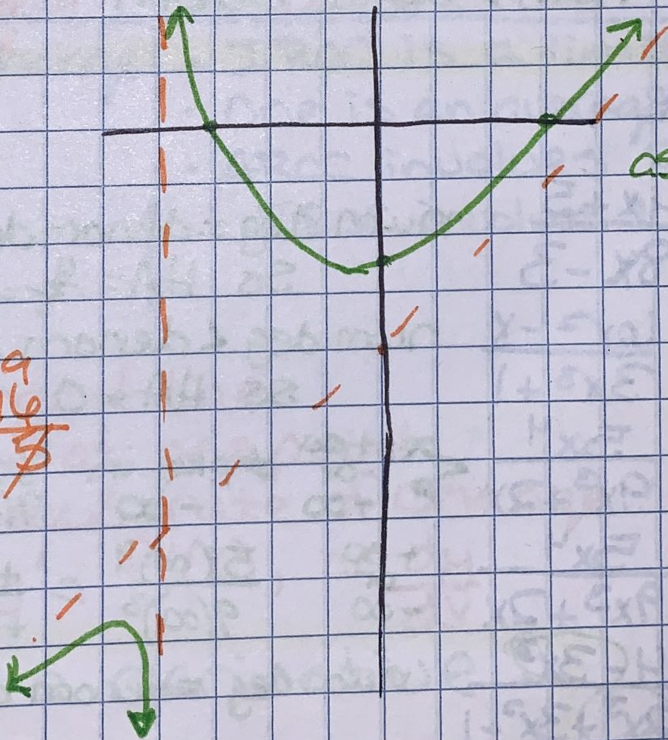
3. $f(x) = \frac{x^2-9}{x+4}$

UA: $x = -4$

HA: none

SA: $y = x - 4$

$$\begin{array}{r|rrr} -4 & 1 & 0 & -9 \\ & \downarrow & -4 & 16 \\ \hline & 1x & -4 & : \cancel{7} \end{array}$$



as $x \to \infty$, $f(x) \to \infty$

$x \to -\infty$, $f(x) \to -\infty$

28. Infinite Limits

Limits of Polynomials at Infinity → follow end behavior rules

1. $\lim_{x \to -\infty} (x^3 - 2x^2 + 5x - 1) = -\infty$ odd positive
2. $\lim_{x \to \infty} (4 + 3x - x^2) = -\infty$ even negative
3. $\lim_{x \to \infty} (5x^4 - 3x) = \infty$ even positive

Limits of Rational Functions → follow Horiz. Asym. rules

- degree num = degree denom.: $\lim_{x \to \pm\infty} f(x) =$ ratio of leading coeff.
- degree num < degree denom: $\lim_{x \to \pm\infty} f(x) = 0$
- degree num > degree denom: $\lim_{x \to \pm\infty} f(x) = +\infty$ or $-\infty$

Evaluate

1. $\lim_{x \to \infty} \frac{4x+5}{8x-3}$ num deg = denom deg. so HA $= \frac{4}{8} = \frac{1}{2}$ $= \frac{1}{2}$
2. $\lim_{x \to -\infty} \frac{6x^2-x}{3x^3+1}$ num deg < denom deg. so HA = 0 $= 0$
3. $\lim_{x \to -\infty} \frac{5x^4}{9x^3+2x}$ $+\infty$ or $-\infty$ *plug in $-\infty$ $\frac{5(-\infty)^4}{9(-\infty)^3} = \frac{+\infty}{-\infty}$ $= -\infty$
4. $\lim_{x \to \infty} \frac{5x^4}{9x^3+2x}$ $+\infty$ or $-\infty$ $\frac{5(\infty)^4}{9(\infty)^3} = \frac{+\infty}{+\infty}$ $= \infty$
5. $\lim_{x \to \infty} \frac{4-3x^3}{2x^3+3x^2-1}$ num deg = denom deg $= -3/2$
6. $\lim_{x \to -\infty} \frac{\sqrt{4x^2+5}}{x-3}$ $\frac{\sqrt{4x^2}}{x} = \frac{|2x|}{x}$ so 2 or −2 same degree $= -2$
7. $\lim_{x \to \infty} \frac{10-3x}{(2x+1)^3}$ num deg < denom. deg. $= 0$