VOLUME OF SOLIDS WITH KNOWN CROSS SECTIONS

Honors Calculus

Keeper 36

VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

For cross sections of area $A(x)$ taken perpendicular to the $x - axis$,

$$Volume = \int_a^b A(x)dx$$

For cross sections of area $A(y)$ taken perpendicular to the $y - axis$

$$Volume = \int_a^b A(y)dy$$

STEPS FOR FINDING VOLUME

1. Draw the base on the xy −plane
2. Draw a representative cross section
3. Find an area formula for the cross section $A(x)$ or $A(y)$ by plugging in the base equation to the area formula of your cross section
4. Set up integral with bounds
5. Integrate your area formula and use FTC on your bounds

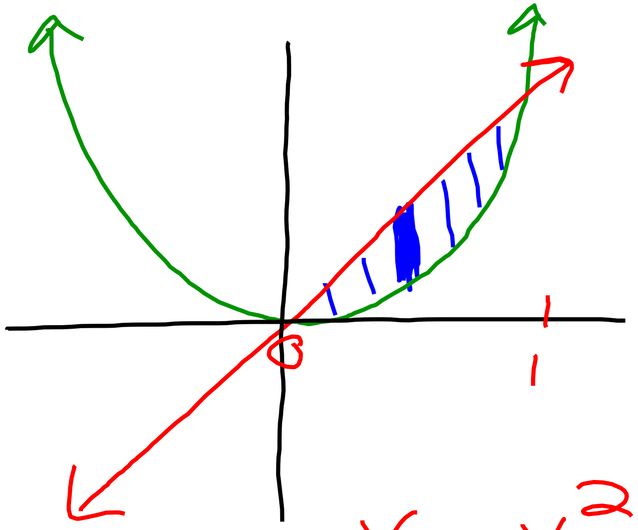
IMPORTANT AREA FORMULAS TO KNOW!

Square	Semicircle	Rectangle	Isosceles Right Triangle with base as leg
$A = b^2$	$A = \frac{1}{2}\pi\left(\frac{b}{2}\right)^2$ Or $A = \frac{\pi}{8}(b)^2$	$A = \mathrm{bh}$	$a = \frac{1}{2}b^2$

VOLUMES BY CROSS SECTIONS LAB

EXAMPLE 1

Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the $x - axis$ are squares.



Top-bottom: $x - x^2 = s$

$A_{square} = (x - x^2)^2 \, dx$

$V = \int_0^1 (x - x^2)^2 \, dx$

$x = x^2$

$0 = x^2 - x$

$0 = x(x - 1)$

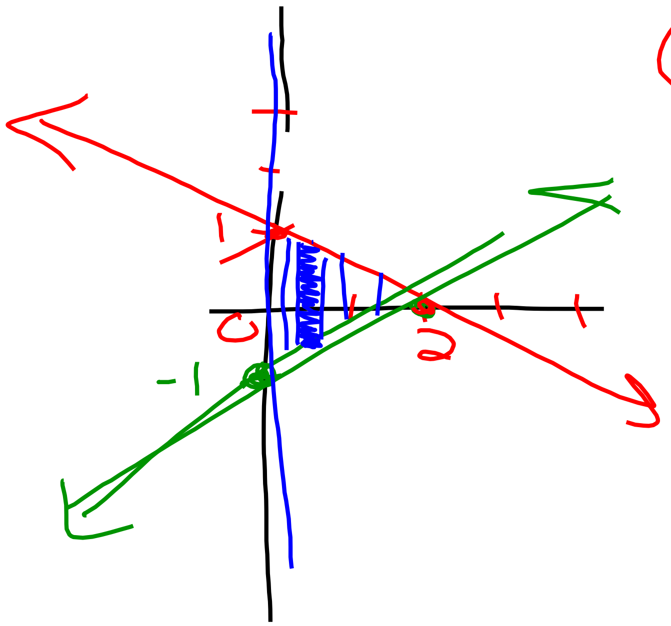
$x = 0 \quad x = 1$

$V = \frac{1}{30}$

EXAMPLE 2

Find the volume of the solid whose base is the region bounded by $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$, *and* $x = 0$. The cross sections are isosceles right triangles with the base as a leg that are perpendicular to the x-axis.

$y = -\frac{1}{2}x + 1$ $\quad$ $y = \frac{1}{2}x - 1$



$$\left(-\frac{1}{2}x+1\right) + \left(-\frac{1}{2}x+1\right) = -x+2 = b$$

$$A = \frac{1}{2}(-x+2)^2\,dx$$

$$V = \frac{1}{2}\int_0^2 (-x+2)^2\,dx$$

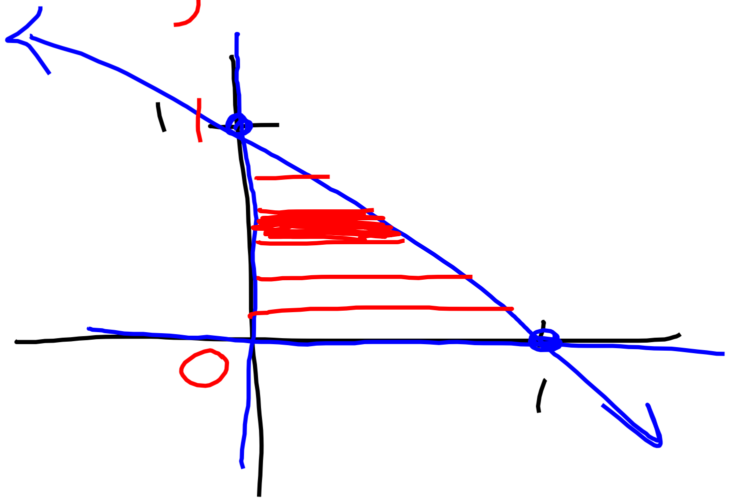
$$V = \frac{1}{2}\int_0^2 (x^2-4x+4) = \frac{1}{2}\left(\frac{x^3}{3} - \frac{4x^2}{2} + 4x\Big|_0^2\right)$$

$$V = \frac{4}{3}$$

EXAMPLE 3

Find the volume of the solid whose base is the triangle enclosed by x + y = 1, the x-axis and the y-axis. Cross sections perpendicular to the y-axis are in the shape of semi-circles.

$y = -x + 1$



$x + y = 1$

$x = 1 - y$

$A = \frac{\pi}{8}(1-y)^2$

$V = \frac{\pi}{8}\int_0^1 (1-y)^2 \, dy$

$V = \frac{\pi}{8}\int_0^1 1 - 2y + y^2 \, dy$

$-\frac{2y^2}{2}$

$\frac{\pi}{8}\left(y - y^2 + \frac{y^3}{3}\right)\Big|_0^1 = \frac{\pi}{8}\left(1 - 1 + \frac{1}{3}\right)$

$V = \frac{\pi}{24}$

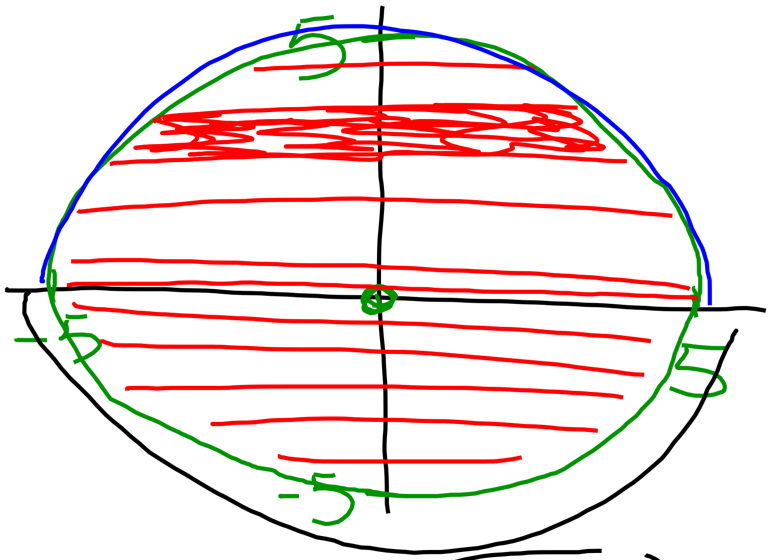
EXAMPLE 4

circle = $(x-h)^2+(y-k)^2=r^2$ center (h,k) r = radius

Find the volume of the solid whose base is the bounded by the circle $x^2 + y^2 = 25$. Cross sections perpendicular to the y-axis are in the shape of squares.

solve so eq. is in terms of y

$x = +\sqrt{25-y^2}$



$X = -\sqrt{25-y^2}$

$$x^2+y^2=25$$
$$x^2=25-y^2$$
$$x=\pm\sqrt{25-y^2}$$

$$S=\sqrt{25-y^2}-\left(-\sqrt{25-y^2}\right)$$
$$S=2\sqrt{25-y^2}$$

$$V=\int_{-5}^{5}\left(2\sqrt{25-y^2}\right)^2 dy$$

$$V=\frac{2000}{3}$$