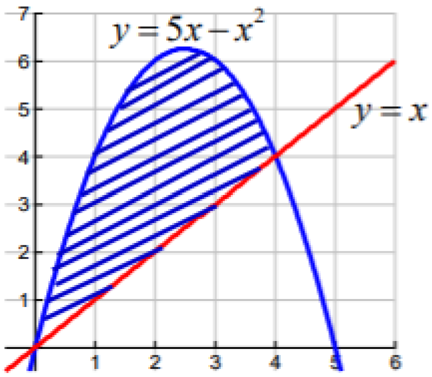
AP Calculus AB: Unit 8 Applications of Integration

Day	Date	Topic	Assignment
1	Friday, Nov. 20th	**Keeper 8.1** – Area Between Two Curves	Area Between Curves (Packet pgs. 1 – 4)
2	Monday, Nov. 30th	**Keeper 8.2** – The Average Value Theorem and Mean Value Theorem	Average Value Theorem (Packet pgs. 5 – 6) Mean Value Theorem (Packet pg. 7)
3	Tuesday, Dec. 1st	**Keeper 8.3** – Volumes of Solids with Known Cross Sections	**Skills Check – Keepers 8.1 – 8.2** Volume with Cross Sections (Packet pg. 8)
4	Wednesday, Dec. 2nd	Optional Q & A Review Keepers 8.1 – 8.3	Catch up on all keeper notes and homework.
5	Thursday, Dec. 3rd	**Keeper 8.4** – Volumes of Revolution (Disk and Washer)	Volumes of Revolution: Disk Method and Washer Method (Packet pgs. 9 - 10)
6	Friday, Dec. 4th	**Keeper 8.5** – Slope Fields	**Skills Check – Keepers 8.3 – 8.4** Slope Fields (Packet pgs. 11 – 12)
7	Monday, Dec. 7th	**Keeper 8.6** – Separable Differential Equations	Differential Equations (Packet pgs. 13 – 14)
8	Tuesday, Dec. 8th	**Keeper 8.7** – Exponential Growth and Decay	**Skills Check – Keepers 8.5 – 8.6** Exponential Growth and Decay (Packet pgs. 15 – 16)
9	Wednesday, Dec. 9th	Optional Q & A Additional Review Unit 8	Complete all keeper notes and homework. Complete Additional Unit 8 Review
10	Thursday, Dec. 10th	Unit 8 Review – Applications of Integration	Complete Unit 8 Homework Packet Complete Unit 8 Additional Review 2
11	Friday, Dec. 11th	Unit 8 Test – Applications of Integration	Begin Studying for the Final Exam

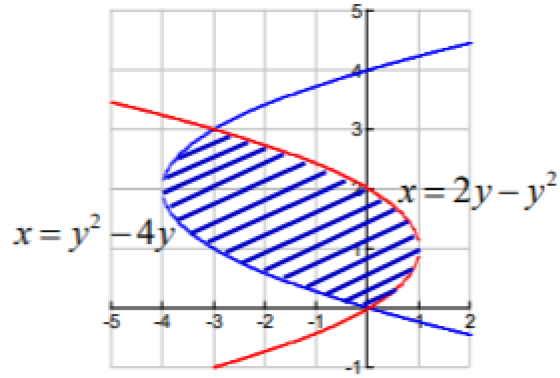
Area Between Curves

Find the area of the shaded region.

1. Calculator

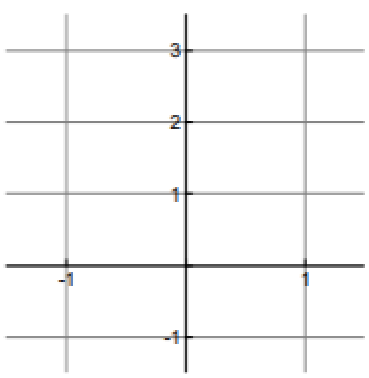


2. Non-Calculator

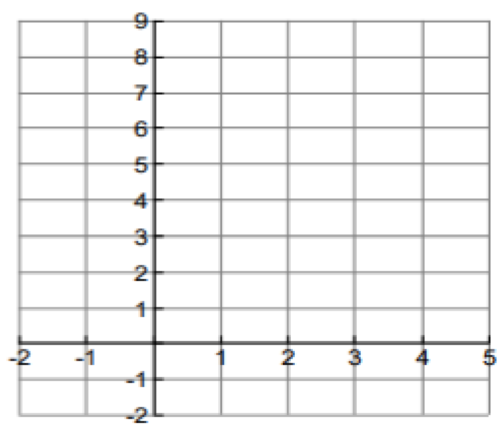


Sketch the region enclosed by the given curves, then approximate.

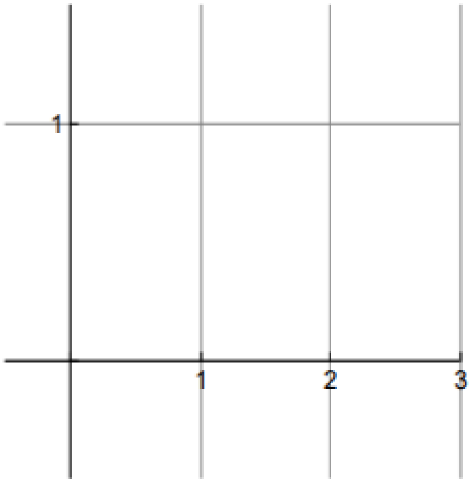
3. $y = e^x, y = x^2 - 1, x = -1, x = 1$ (Calculator)



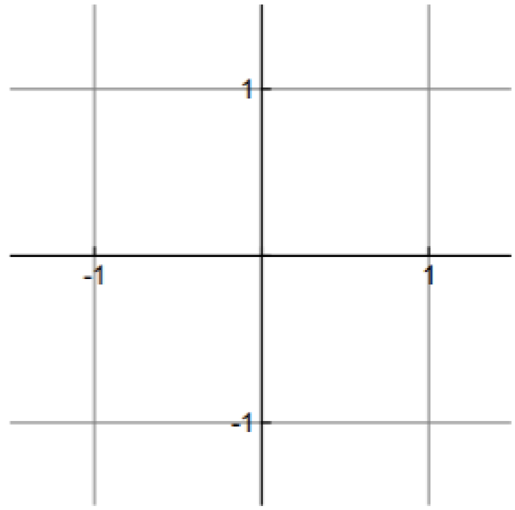
4. $y = x^2 - 2x, y = x + 4$ (Non-Calculator)



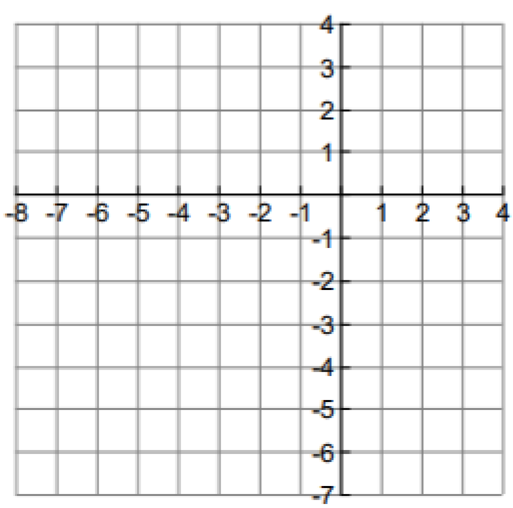
5. $y = \frac{1}{x}, y = \frac{1}{x^2}, x = 2$ (Calculator)



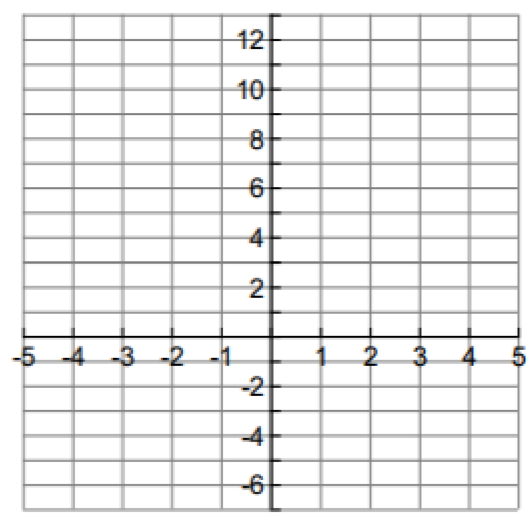
6. $x = 1 - y^2, x = y^2 - 1$ (Non-Calculator)



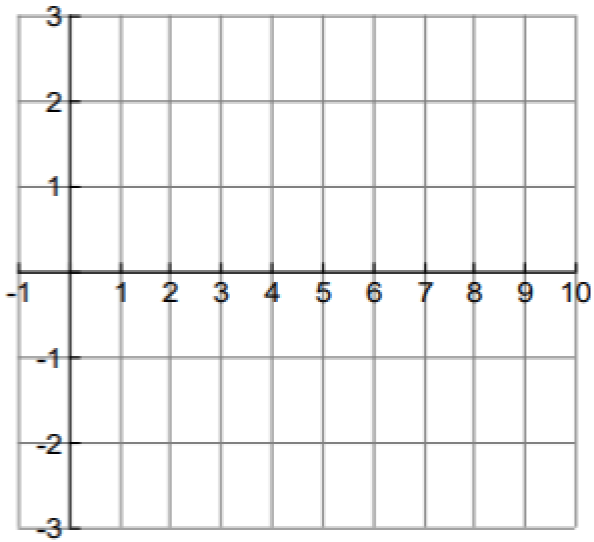
7. $4x + y^2 = 12,\ x = y$ (Calculator)



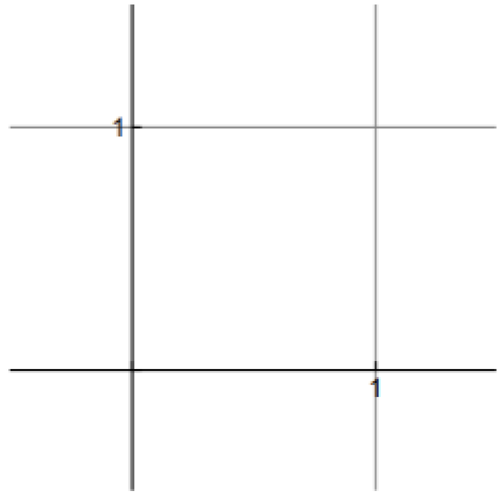
8. $y = 12 - x^2, y = x^2 - 6$ (Calculator)



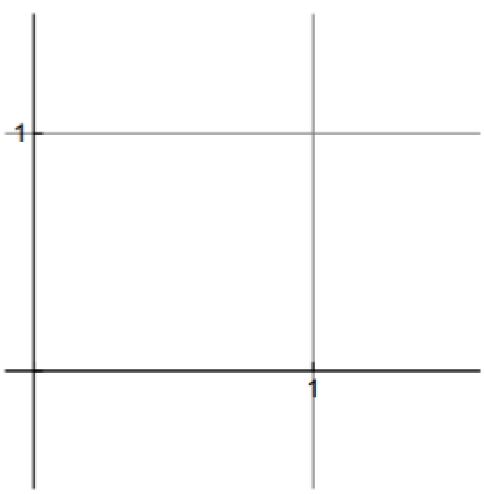
9. $x = 2y^2, x = 4 + y^2$ (Calculator)



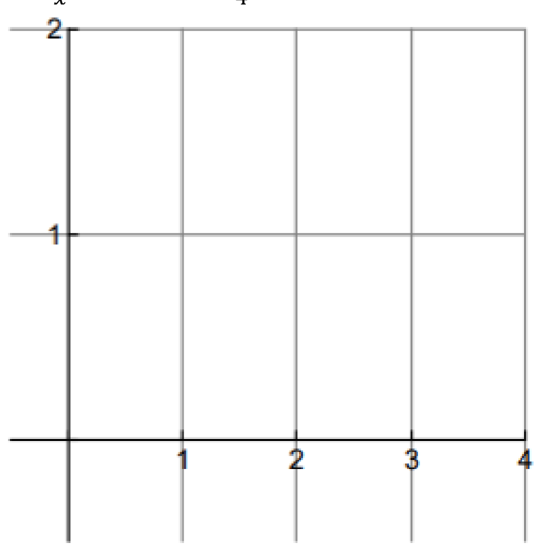
10. $y = x^2, y = x$ (Non-Calculator)



11. $y = \cos x\,, y = \sin 2x\,, x = 0, x = \frac{\pi}{2}$ (Calculator)



12. $y = \frac{1}{x}, y = x, y = \frac{1}{4}x, x > 0$ (Calculator)



Compute the area of the region which is enclosed by the given curves.

13. $y = 4x,\ y = 6x^2$

14. $y = 2x^2, y = x^2 + 2$

15. $y = x^{\frac{2}{3}},\ y = x^4$, in the first quadrant

16. $y = \frac{1}{x},\ y = \frac{1}{x^2},\ x = 4$

17. $y = \sin x\,,\ y = 2 - \sin x\,, \frac{\pi}{2} \le x \le \frac{5\pi}{2}$

18. $y = e^{5x},\ y = e^{8x}, x = 1$

Average Value Theorem

Find the average value of the function on the given interval.

1. $f(x) = 4x - x^2,\ [0,4]$

2. $f(x) = \sin(4x),\ [-\pi, \pi]$

3. $g(x) = \sqrt[3]{x},\ [1,8]$

4. $f(t) = e^{\sin t} \cos t,\ \left[0, \frac{\pi}{2}\right]$

5. $h(x) = \cos^4 x \cdot \sin x,\ [0, \pi]$

6. $h(u) = (3 - 2u)^{-1},\ [-1,1]$

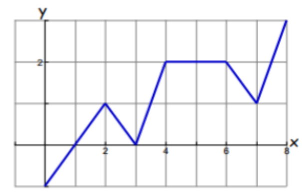
A. Find the average value of f on the given interval. B. Find c such that $f_{avg} = f(c)$.

7. $f(x) = (x - 3)^2,\ [2,5]$

8. $f(x) = \frac{1}{x},\ [1,3]$

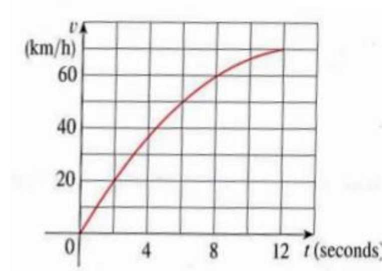
9. Find the number b, such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

10. Find the average value of f on $[0,8]$



11. The velocity graph of an accelerating car is shown.

a. Use the Midpoint rule to estimate the average velocity of the car during the first 12 seconds.



b. At what time was the instantaneous velocity equal to the average velocity?

12. In a certain city the temperature (in $°F$) t hours after 9 am was modeled by the function $T(t) = 50 + 14\sin\left(\frac{\pi t}{12}\right)$. Find the average temperature during the period from 9 am to 9 pm.

Mean Value Theorem

For each problem, find the values of c that satisfy the Mean Value Theorem for Integrals. Set up the integral and use the calculator to solve.

11. $f(x) = -\frac{x^2}{2} + x + \frac{3}{2};\ [-3,1]$

12. $f(x) = \frac{4}{x^2};\ [-4,-2]$

13. $f(x) = 4\sqrt{x};\ [0,3]$

14. $f(x) = \frac{1}{x};\ [2,3]$

15. $f(x) = x^5 - 2x^3 + x; [-1,0]$

16. $f(x) = x^5 - 4x^3 + 2x - 1;\ [-2,2]$

17. $f(x) = -x + 2;\ [-2,2]$

18. $f(x) = \frac{4}{(2x+6)^2};\ [-6,-5]$

19. $f(x) = -x^2 - 8x - 17;\ [-6,3]$

20. $f(x) = -3(2x-6)^{\frac{1}{2}};\ [3,5]$

Volumes with Cross Sections

1. The base of a solid is bounded by $y = \cos(x)$, the x-axis, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Cross sections perpendicular to the x-axis are squares. Find the volume.

2. The base of a solid is bounded by $y = 2 - x$, the x-axis, and the y-axis. Cross sections that are perpendicular to the x-axis are isosceles right triangles with the right angle on the x-axis. (Legs perpendicular to the x-axis). Find the volume.

3. The base of a solid is bounded by the semi-circle $y = \sqrt{4 - x^2}$ and the x-axis. Cross sections that are perpendicular to the x-axis are squares. Find the volume.

4. The base of a solid is bounded by $y = \sqrt{16 - x^2}$ and the x-axis. Cross sections that are perpendicular to the y-axis are equilateral triangles. Find the volume.

5. The base of a solid is a circular region in the xy-plane bounded by the graph $x^2 + y^2 = 9$. Find the volume of the solid if every cross section by a plane normal to the x-axis is an equilateral triangle with one side as the base.

6. The base of a solid is circular region in the xy-plane bounded by the graph of $x^2 + y^2 = 9$. Find the volume of the solid if every cross section by a plane normal to the x-axis is a square with one side as the base.

7. The base of a solid is bounded by $y = 2 - \frac{1}{2}x$, the x-axis, and the y-axis. Cross sections that are perpendicular to the y-axis are isosceles right triangles with the hypotenuse in the xy-plane. Find the volume.

Volumes of Revolution: Disk Method

Find the Volumes of Revolution:

1. $y = \sqrt{x}, x = 1, x = 4, y = 0$ about the x-axis

2. $y = -x + 1, y = 0, x = 0$ about the x-axis

3. $y = 4 - x^2, y = 0, x = 0$, (in the 1st quadrant) about the x-axis

4. $y = x^2, x = 0, y = 4$, (in the 1st quadrant) about the y-axis

5. $y = \sqrt{4 - x^2}, y = 0, x = 0$, (in the 1st quadrant) about the x-axis

6. $x = 4y - y^2, y = 1, x = 0$, about the y-axis

7. $y = x^{\frac{2}{3}}, y = 1, x = 0$, about the y-axis

8. $y = 5x - x^2, y = 0$, about the x-axis

9. $y = \frac{x^2}{2}, y = 8$, about the line $y = 8$

10. $x = \sqrt{y}, x = 9, y = 0$, about $x = 9$

Volumes of Revolution: Washer Method

Find the Volumes of Revolution:

1. $f(x) = 2\sqrt{x}\ and\ g(x) = x^2$ about the x-axis

2. $f(x) = 2\sqrt{x}\ and\ g(x) = x^2$ about the y-axis

3. $f(x) = 2\sqrt{x}\ and\ g(x) = x^2$ about the line $y = 3$

4. $f(x) = 2\sqrt{x}\ and\ g(x) = x^2$ about the line $y = -1$

5. $y = x^2 + 1, y = 0, x = 1,\ x = 0$ about the y-axis

6. $y = \frac{1}{x}, y = 2, and\ x = 2$ about the y-axis

7. $y = x, y = 2 - x^2, and\ x = 0$ about the x-axis

8. $y = x^2\ and\ \ y = 2x$, about the y-axis

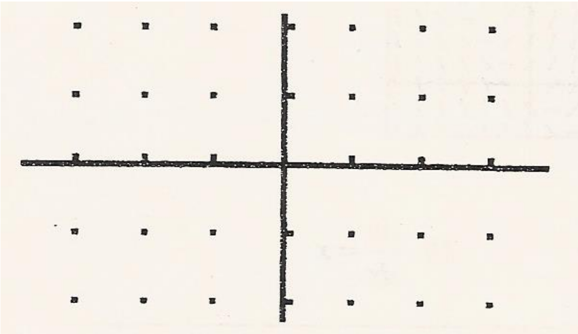
9. $y = x^2, and\ y = x + 2$, about the $x - axis$

10. $y = 2x + 2\ and\ y = x^2 + 2$ about the x-axis

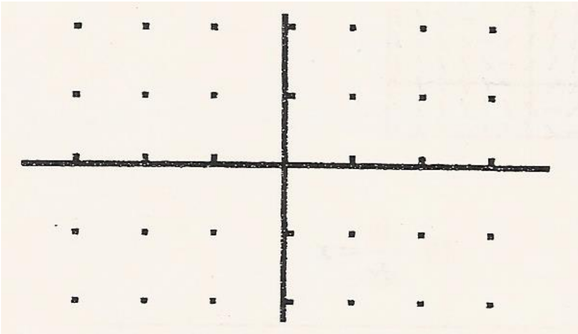
SLOPE FIELDS

Draw a slope field for each of the following differential equations.

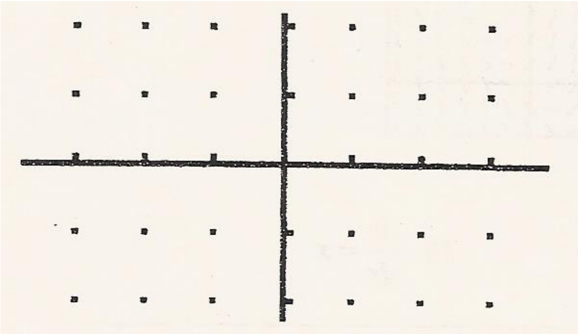
1. $\frac{dy}{dx} = x+1$



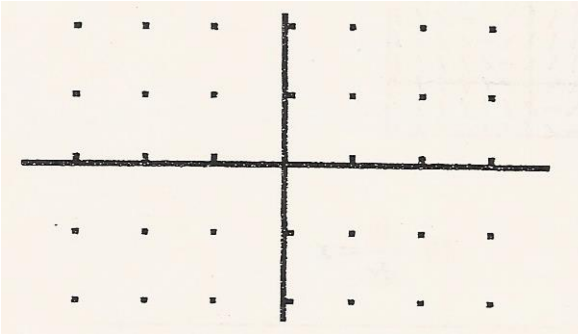
2. $\frac{dy}{dx} = 2y$



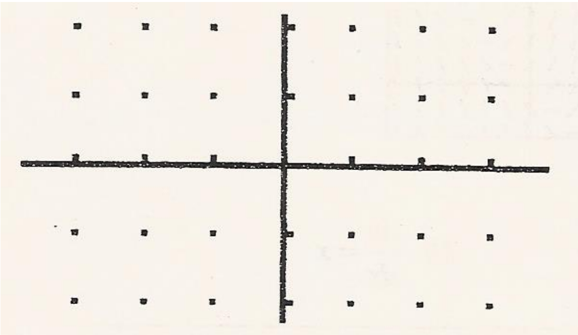
3. $\frac{dy}{dx} = x+y$



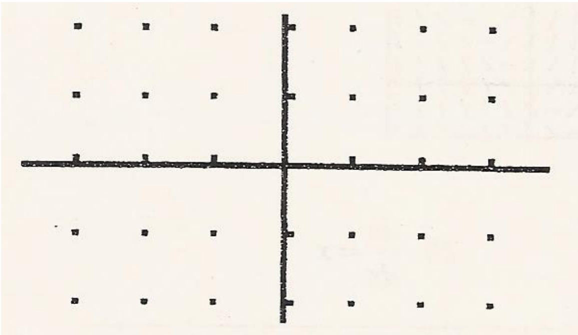
4. $\frac{dy}{dx} = 2x$



5. $\frac{dy}{dx} = y-1$

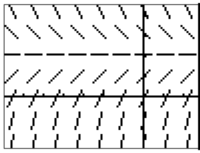


6. $\frac{dy}{dx} = -\frac{y}{x}$

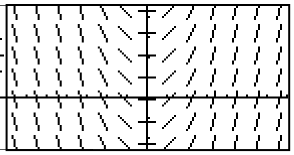


Match the slope fields with their differential equations.

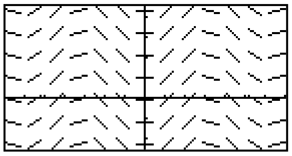
(A)



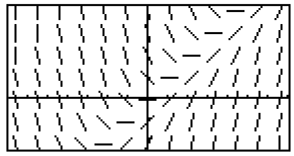
(B)



(C)



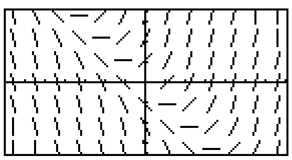
(D)



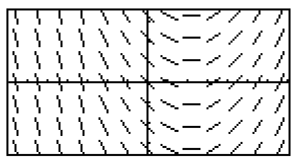
7. $\frac{dy}{dx} = \sin x$ 8. $\frac{dy}{dx} = x - y$ 9. $\frac{dy}{dx} = 2 - y$ 10. $\frac{dy}{dx} = x$

Match the slope fields with their differential equations.

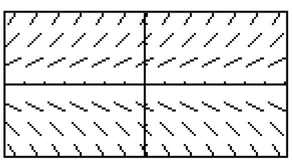
(A)



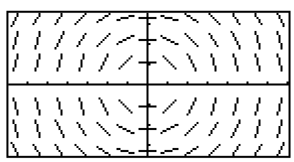
(B)



(C)



(D)



11. $\frac{dy}{dx} = .5x - 1$ 12. $\frac{dy}{dx} = .5y$ 13. $\frac{dy}{dx} = -\frac{x}{y}$ 14. $\frac{dy}{dx} = x + y$

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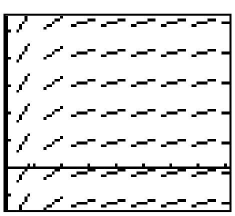
15. (From the AP Calculus Course Description)



The slope field from a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?

(A) $y = x^2$ (B) $y = e^x$ (C) $y = e^{-x}$ (D) $y = \cos x$ (E) $y = \ln x$

Differential Equations

Solve the differential equation.

1. $\frac{dy}{dx} = xy^2$

2. $\frac{dy}{dx} = xe^{-y}$

3. $xy^2y' = x + 1$

4. $(y^2 + xy^2)y' = 1$

5. $(y + \sin y)y' = x + x^3$

6. $\frac{dp}{dt} = t^2p - p + t^2 - 1$

Find the solution to the differential equation that satisfies the given initial condition.

8. $\frac{dy}{dx} = \frac{x}{y}, y(0) = -3$

9. $\frac{dy}{dx} = \frac{\ln x}{xy}, y(1) = 2$

10. $\frac{du}{dt} = \frac{2t+\sec^2 t}{2u}, u(0) = -5$

11. $\frac{dP}{dt} = \sqrt{Pt}, P(1) = 2$

12. $\frac{dy}{dx} = 6x^2 + 6x + 2$ and $f(-1) = 2$

13. $\frac{dy}{dx} = \frac{1+12x^{3/2}}{2\sqrt{x}}$ and $f(0) = 2$

Exponential Growth and Decay

Ex 1: Modeling Penicillin Pharmacologists have shown that the rate at which penicillin leaves a person's bloodstream is proportional to the amount of penicillin present.

(a) Express this statement as a differential equation.

(b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours after an initial injection of 450 mg.

(c) At what time was 200 mg of penicillin present?

Ex 2: Computing doubling time Some studies have suggested that from 1955 to 1970, the number of bachelor's degrees in physics awarded per year by U.S. universities grew exponentially, with growth constant k = 0.1 (approximately 2,500 degrees awarded in 1955).

(a) What was the doubling time?

(b) How long would it take from the number of degrees awarded per year to increase 8-fold?

Ex 3: One of the world's smallest flowering plants, *Wolffia globosa*, has a doubling time of approximately 30 hours. Find the growth constant k and determine the initial population if the population grew to 1,000 after 48 hours.

Ex 4: A principal of $10,000 are deposited into an account paying 6% interest. Find the balance after 3 years if (a) the interest is compounded quarterly and (b) if interest is compounded

Ex 5: A certain bacteria population P obeys the exponential growth law $P(t) = 2,000e^{1.3t}$ where t is in hours.

(a) How many bacteria are present initially?

(b) At what time will there by 10,000 bacteria?

Ex 6: A certain RNA molecule replicates every 3 minutes. Find the differential equation for the number N(t) of molecules present at time t (in minutes). Starting with one molecule, how many will be present after 10 min?

Ex 7: The decay constant of Cobalt-60 is 0.13 years^{-1}. What is its half-life?

Ex 8: Find the decay constant of Radium-226, given that its half-life is 1,622 years.

Ex 9: The population of Washington state increased from 4.86 million in 1990 to 5.89 million in 2000.

(a) What will the population be in 2010?

(b) What is the doubling time?

Ex 10: An insect population triples in size after 5 months. When will it quadruple is size?

Ex 11: A 10-kg quantity of a radioactive isotope decays to 3 kg after 17 years. Find the decay constant of the isotope.

Ex 12: The isotope Thorium-234 has a half-life of 24.5 days.

(a) Find the differential equation satisfied by the amount y(t) of Thorium-234 in a sample at time t.

(b) At t = 0, a sample contains 2 kg of Thorium-234. How much remains after 1 year?

Ex 13: After four days a sample of radon-222 decayed to 45% of its original amount. Radon-222 decays at a rate proportional to the amount present.

a. Find the growth model for the amount of radon-222 present at time t.

b. What is the half-life of radon-222?

c. How long would it take the sample to decay to 20% of its original amount?

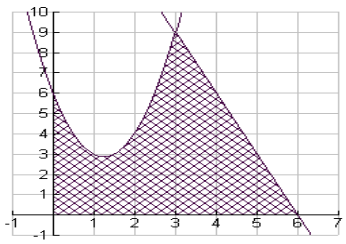
Integral Applications Practice Test

1. **Multiple choice:** The average value of f(x) = $\sec^2 x$ over the interval $0 \le x \le \frac{\pi}{4}$ is

   a) $\frac{2\sqrt{2}}{\pi}$ b) $\frac{\pi}{4}$ c) $\frac{4}{\pi}$ d) 1 e) none of these

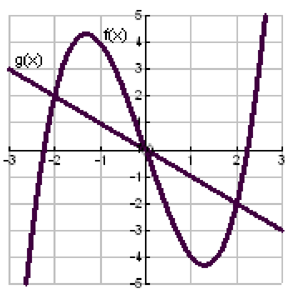
2. Find the average value of $f(x) = \frac{4x}{e^{x^2}}$ on [0, 3].

3. ***Set up only*** the integral needed to find the area bounded by
   $y = 2x^2 - 5x + 6$, $y = -3x + 18$, y = 0, and x = 0.



4. ***Set up only:*** Find the area bound by the graphs of $x = y^2 - 4$ and x = y − 2.

5. **Set up only.** Sketch the graph and set up the integral used to find the area between the curves
f(x) = x^3-5x and g (x) = -x.

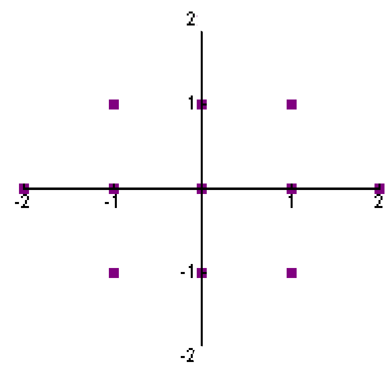


6. Cross sections perpendicular to the y-axis are equilateral triangles. Find the volume of the solid formed by these cross sections and bound by $9x^2 + 4y^2 = 36$. ***Set up only.***

7. Semicircles are stacked perpendicular to the **x-axis** on the base in the first quadrant determined by $y = x^3$, $y = 4$, and $x = 0$. ***Set up*** the integral used to find the volume of the solid generated by these cross sections.

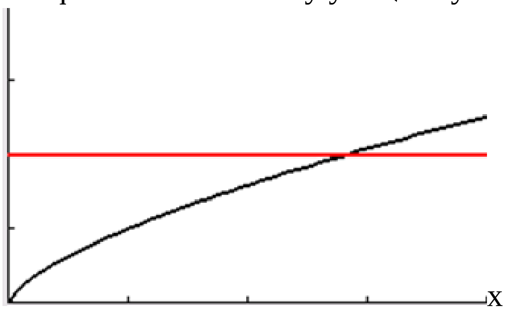
8. Consider the differential equation $\frac{dy}{dx} = (2 + y)x.$

a) On the axes provided, sketch a slope field for the given differential equation at the eleven points indicated. Sketch the solution to the differential equation that passes through the point (0, 0).



b) Find the solution to the differential equation above with initial condition y(0)=3.

9. ***Set up only.*** Set up the integral needed to find the volume of revolution formed when the region in the first quadrant bounded by y = $\sqrt[3]{x^2}$ y = 2, and x = 0 is revolved around the indicated axis.



a) about the x-axis

b) about the line x = - 1 (use shell method)

c) about the y-axis

d) about the line y = -2

10. Find a particular solution of the differential equation $\frac{dy}{x} = \sin(x^2)dx$ with initial condition y(0)= -1.

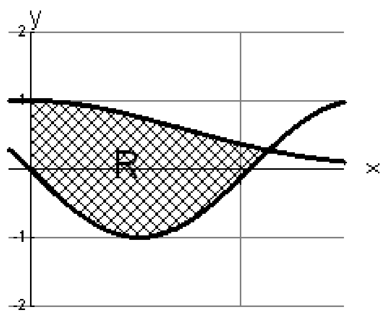
11. Find the particular solution to the differential equation.

$\frac{dP}{dt} = 3P - 4Pt$ if when t = 0, P = 6.

CALCULATOR SECTION

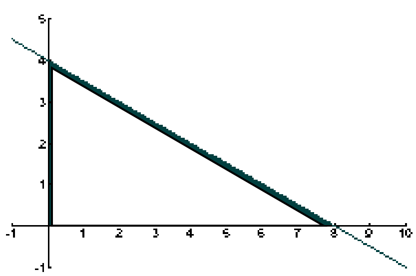
12. **Multiple Choice.** Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = -\sin(3x)$, and the y-axis as shown in the figure below. Which of the following gives the approximate area of the region R? (Show what integral you set up and solve).

A) 1.139 (B) 1.445 (C) 1.869 (D) 2.114 (E) 2.340



13. The base of a solid is a region in the first quadrant bounded by the x-axis, the y-axis, and the line x + 2y = 8, as shown below. If cross sections of the solid perpendicular to the x-axis are isosceles right triangles set on a leg, what is the volume of the solid?

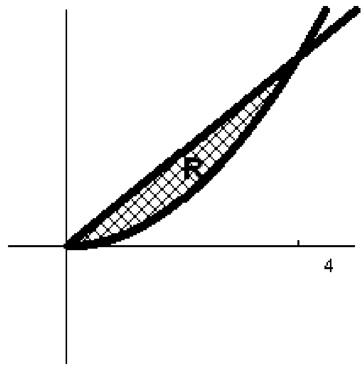
(A) 10.667 (B) 14.661 (C) 16.755 (D) 21.333 (E) 42.667



14. The area under the curve $y = \frac{1}{x}$ from x = 1 to x = k is 1. Find the value of k.

15. Let R be the shaded region bounded by the graph of y = x^2 and the line y = 4x as shown.

A) Set up and evaluate the integral needed to find the area between the functions.



B) ***Set up*** the integral to find the volume if region R is revolved about the line x = −2.

16. The volume V in liters of air in the lungs during a five-second respiratory cycle is approximated by the model

$$V = 0.1729t + 0.1522t^2 - 0.0374t^3$$

where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

17. Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. Use the fact that the half-life of Plutonium is 24,100 years.

a) Write the growth model.

b) How long will it take for the 10 grams to decay to 1 gram?