Average Function Value for Integrals (7.1)

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Find the average value of the function over the given interval.

1. $f(x) = -x^2+2x+1 \quad [\overset{a}{1},\overset{b}{4}]$

$$f_{ave} = \frac{1}{4-1}\int_1^4 -x^2+2x+1\,dx$$

$$\frac{1}{3}\left(\frac{-x^3}{3}+x^2+x\Big|_1^4\right) = \frac{1}{3}\left[\left(\frac{-64}{3}+16+4\right)-\left(\frac{-1}{3}+1+1\right)\right]$$

$$\frac{1}{3}[-3] = -1$$

2. $f(x) = -2e^{2x+4} \quad [-3,-2]$

$$f_{ave} = \frac{1}{-2+3}\int_{-3}^{-2} -2e^{2x+4}\,dx$$

$u = 2x+4$
$du = 2\,dx$
$dx = \frac{du}{2}$

$u_{-2} = 0$
$u_{-3} = -2$

$$-\cancel{2}\int_{-3}^{-2} e^u \frac{du}{\cancel{2}}$$

$$-e^u\Big|_{-2}^{0} = -e^0 - -e^{-2} = -1+\frac{1}{e^2}$$

3. $f(x) = \csc^2 x \quad [\pi/2, 3\pi/4]$

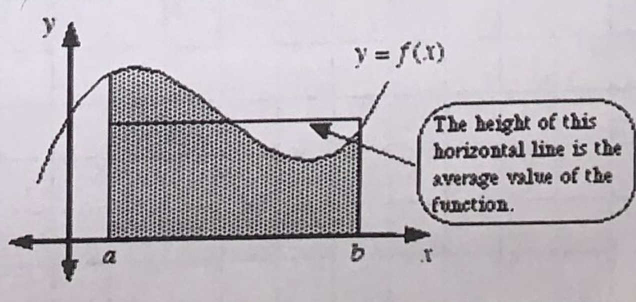
$$\frac{1}{3\pi/4-\pi/2}\int_{\pi/2}^{3\pi/4}\csc^2 x\,dx$$

$$\frac{1}{\pi/4}\int_{\pi/2}^{3\pi/4}\csc^2 x\,dx = \frac{4}{\pi}\left(-\cot x\Big|_{\pi/2}^{3\pi/4}\right)$$

$$\frac{4}{\pi}\left(-\cot\frac{3\pi}{4} - -\cot\frac{\pi}{2}\right) = \frac{4}{\pi}\left(-(-1)+0\right) = \frac{4}{\pi}$$

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The rectangle has the same area as the shaded region under the curve.



82 Mean Value Theorem for Integrals

$$\boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$$

There is a rectangle whose area is precisely equal to the area of the region under the curve of the entire interval.

Find the value of c guaranteed by the MVT.

1. $f(x) = \frac{5}{x^2}$ $[1,4]$

$$\int_1^4 \frac{5}{x^2}dx = \frac{5}{c^2}(4-1)$$

$$\left.\frac{-5}{x}\right|_1^4 = \frac{15}{c^2} \quad = \frac{-5}{4} - -5 = \frac{15}{c^2}$$

$$\frac{15}{4} = \frac{15}{c^2}$$

$$c^2 = 4$$

$$c = \pm 2$$

$c = 2$ only by -2 isn't in the interval $[1,4]$

2. $f(x) = -x+1$ $[-6,-5]$

$$\int_{-6}^{-5} -x+1\,dx = (-c+1)(-5--6)$$

$$\left.-\frac{x^2}{2}+x\right|_{-6}^{-5} = -c+1$$

$$\left(\frac{-25}{2}-5\right) - \left(\frac{-36}{2}-6\right) = -c+1$$

$$6.5 = -c+1$$

$c = -5.5$

3. $f(x) = 3x^{1/2}$ $[0,9]$

$$\int_0^9 3x^{1/2}dx = 3c^{1/2}(9-0)$$

$$\left.2x^{3/2}\right|_0^9 = 27c^{1/2}$$

$$2\sqrt{9}^3 - 2\sqrt{0}^3 = 27\sqrt{c}$$

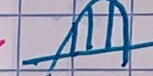
$$54 = 27\sqrt{c}$$

$$2 = \sqrt{c}$$

$c = 4$

Area Between Two Curves (7.2)

$A = \int_a^b [f(x) - g(x)]\,dx$

Top - Bottom → dx 

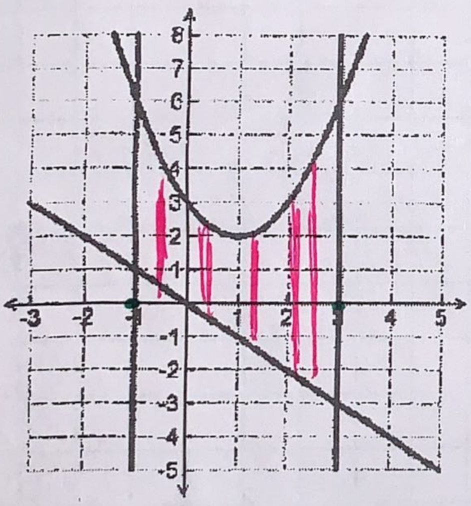
or

Right - Left → dy 

Find the area of the region bounded by the graphs.

1. $f(x) = x^2 - 2x + 3$

$g(x) = -x$



Top - bottom

$A = \int_{-1}^{3} [(x^2 - 2x + 3) - (-x)]\,dx$

$A = \int_{-1}^{3} x^2 - x + 3\,dx$

$A = \frac{x^3}{3} - \frac{x^2}{2} + 3x \Big|_{-1}^{3}$

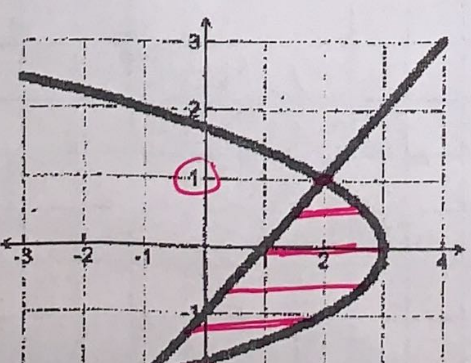
$A = \left(\frac{3^3}{3} - \frac{3^2}{2} + 3(3)\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2} + 3(-1)\right)$

$A = 9 - \frac{9}{2} + 9 + \frac{1}{3} + \frac{1}{2} + 3$

$A = 52/3$

2. $x = 3 - y^2$

$y = x - 1$ $x = y + 1$



Right - Left dy (solve for x)

$A = \int_{-2}^{1} [(3 - y^2) - (y + 1)]\,dy$

$A = \int_{-2}^{1} -y^2 - y + 2\,dy$

$A = -\frac{y^3}{3} - \frac{y^2}{2} + 2y \Big|_{-2}^{1}$

$A = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)$

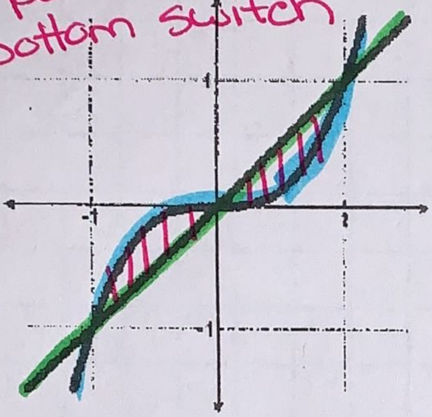
$A = 9/2$ or 4.5

3.

$f(x) = x^3$

$g(x) = x$

2 parts bc top + bottom switch



$$A = \int_{-1}^{0} x^3 - x \, dx + \int_{0}^{1} x - x^3 \, dx$$

$$A = \left(\frac{x^4}{4} - \frac{x^2}{2} \Big|_{-1}^{0}\right) + \left(\frac{x^2}{2} - \frac{x^4}{4} \Big|_{0}^{1}\right)$$

$$A = \left[(0-0) - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{1}{2} - \frac{1}{4}\right) - (0-0)\right]$$

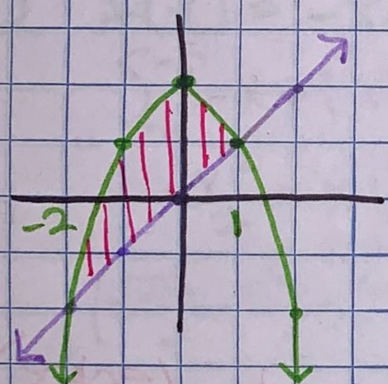
$$A = \frac{1}{4} + \frac{1}{4}$$

$$A = \frac{1}{2}$$

4. $f(x) = 2 - x^2$; $g(x) = x$

① Sketch to decide top/bottom or right/left

② Find pts of intersection by setting $f(x) = g(x)$ + Solve



$2 - x^2 = x$

$0 = x^2 + x - 2$

$0 = (x+2)(x-1)$

$x = -2 \quad x = 1$

$$A = \int_{-2}^{1} 2 - x^2 - x \, dx$$

$$A = -\frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{-2}^{1}$$

$$\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)$$

$$A = \frac{9}{2} \text{ or } 4.5$$

5. $f(x) = \sin x$; $g(x) = \cos x$ Find the area of 1 of the repeated regions.

$$A = \int_{\pi/4}^{5\pi/4} \sin x - \cos x \, dx$$

$$A = -\cos x - \sin x \Big|_{\pi/4}^{5\pi/4}$$

$$A = \left(-\cos\tfrac{5\pi}{4} - \sin\tfrac{5\pi}{4}\right) - \left(-\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4}\right)$$

$$A = \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right) - \left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\right)$$

$$A = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} + \frac{\cancel{2}\sqrt{2}}{\cancel{2}}$$

$$A = 2\sqrt{2}$$

6. $f(x) = 3x^3 - x^2 - 10x$; $g(x) = -x^2 + 2x$

$$3x^3 - x^2 - 10x = -x^2 + 2x$$
$$3x^3 - 12x = 0$$
$$3x(x^2 - 4) = 0$$
$$3x(x+2)(x-2) = 0$$
$$x = 0 \quad x = -2 \quad x = 2$$

$$A = \int_{-2}^{0} (3x^3 - x^2 - 10x) - (-x^2 + 2x) \, dx$$

$$A = \int_{-2}^{0} 3x^3 - 12x \, dx$$

$$A = \frac{3x^4}{4} - 6x^2 \Big|_{-2}^{0}$$

$$A = (0 - 0) - (12 - 24) \qquad A = 12 \text{ (left)}$$

$$A = \int_{0}^{2} (-x^2 + 2x) - (3x^3 - x^2 - 10x) \, dx$$

$$A = \int_{0}^{2} -3x^3 + 12x \, dx$$

$$A = -\frac{3x^4}{4} + 6x^2 \Big|_{0}^{2}$$

$$A = (-12 + 24) - (0 + 0) \qquad A = 12 \text{ (right)}$$

$$A = 12 + 12 = 24$$

Volumes of Solids with Known Cross Sections

7.3

Cross Sections of area taken $\perp$ to X-axis

Volume $= \int_a^b A(x)\,dx$

Cross Sections of area taken $\perp$ to y-axis.

Volume $= \int_a^b A(y)\,dy$

Steps for finding volume:

1. Draw the base on the xy plane.
2. Draw a representative cross section.
3. Find an area formula for the cross section $A(x)$ or $A(y)$ by plugging in the base equation to the area formula of your cross section.
4. Set up integral with bounds
5. Integrate your area formula & use FTC on your bounds.

Important Area Formulas to Know!

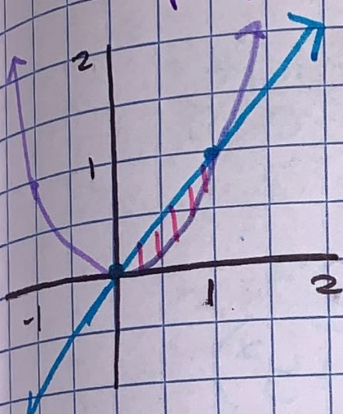
Square	Semicircle	Isosceles Right Triangle w/ base as a Leg
$A = b^2$	$A = \frac{1}{2}\pi\left(\frac{b}{2}\right)^2$ or $A = \frac{\pi}{8}b^2$	$A = \frac{1}{2}b^2$

Equilateral Triangle

$A = \frac{\sqrt{3}}{4}y^2$

continue...

Ex. 1: Find the volume of the solid whose base is the region bounded between the curves $y=x$ and $y=x^2$, and whose cross sections perpendicular to the x-axis are squares.

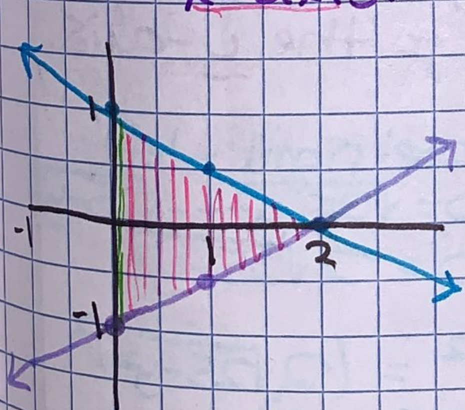


base = top - bottom
$b = x - x^2$

$A = b^2$
$A = (x-x^2)^2$
$A = x^2 - 2x^3 + x^4$

$V = \int_0^1 x^2 - 2x^3 + x^4 \, dx$
$V = \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \Big|_0^1$
$V = \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) - (0 - 0 + 0)$
$V = \frac{1}{30}$

Ex 2: Find the volume of the solid whose base is the region bounded by $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$ and $x = 0$. The cross sections are isosceles right triangles with the base as a leg that are perpendicular to the x-axis.



$f(x) = -\frac{1}{2}x + 1$
$g(x) = \frac{1}{2}x - 1$

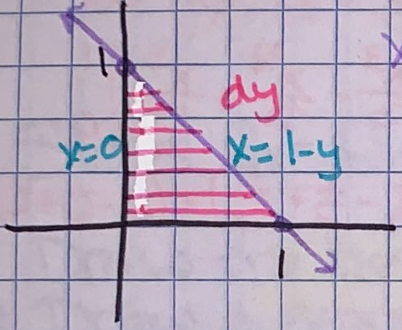
base = top - bottom
$b = \left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right)$
$b = 2 - x$

$A = \frac{1}{2} b^2$
$A = \frac{1}{2}(2-x)^2$
$A = \frac{1}{2}(4 - 4x + x^2) \rightarrow A = 2 - 2x + \frac{x^2}{2}$

$V = \int_0^2 \left(2 - 2x + \frac{x^2}{2}\right) dx$
$V = 2x - x^2 + \frac{x^3}{6} \Big|_0^2$
$V = \left(4 - 4 + \frac{4}{3}\right) - (0 - 0 + 0)$

$V = \frac{4}{3}$

Ex 3: Find the volume of the solid whose base is the triangle enclosed by $x+y=1$, the x-axis & the y-axis. Cross sections perpendicular to the y-axis are in the shape of semicircles.



$x+y=1$
$y=-x+1$

right − left
base: $1-y-0$

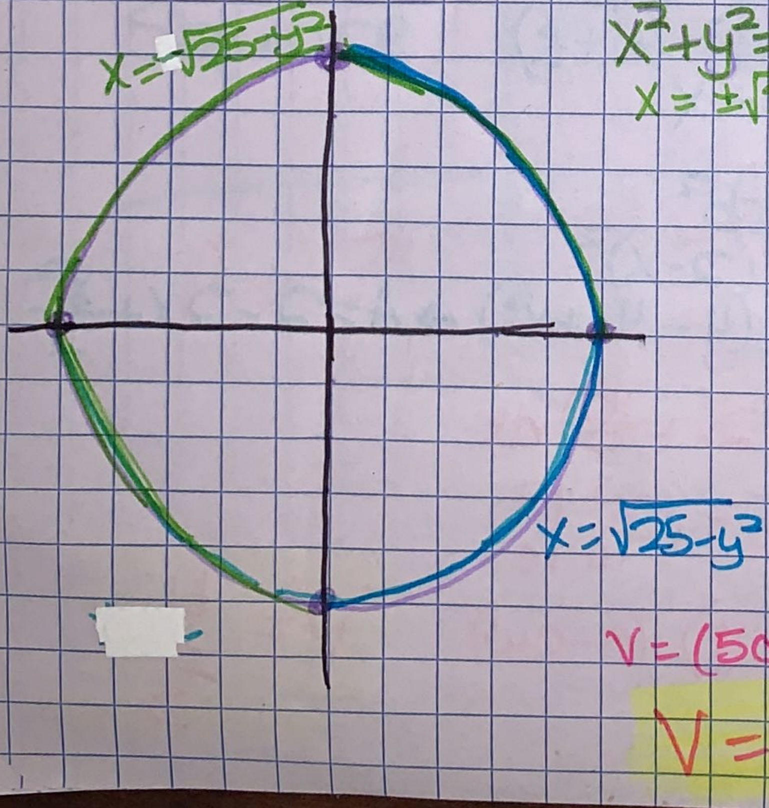
$A=\frac{\pi}{8}b^2$
$A=\frac{\pi}{8}(1-y)^2$
$A=\frac{\pi}{8}(1-2y+y^2)$

$V=\frac{\pi}{8}\int_0^1 (y^2-2y+1)\,dy$

$V=\frac{\pi}{8}\left(\frac{y^3}{3}-y^2+y\Big|_0^1\right)$

$V=\frac{\pi}{8}\left(\frac{1}{3}-1+1\right)$ $\quad V=\frac{\pi}{24}$

Ex 4: Find the volume of the solid whose base is bounded by the circle $x^2+y^2=25$ with square cross-sections $\perp$ to the y-axis.



$x^2+y^2=25$
$x=\pm\sqrt{25-y^2}$

base: right − left
$b=\sqrt{25-y^2}--\sqrt{25-y^2}$
$b=2\sqrt{25-y^2}$

$A=b^2=\left(2\sqrt{25-y^2}\right)^2$
$A=4(25-y^2)$
$A=100-4y^2$

$V=\int_{-5}^{5} 100-4y^2\,dy$

$V=100y-\frac{4y^3}{3}\Big|_{-5}^{5}$

$V=\left(500-\frac{500}{3}\right)-\left(-500+\frac{500}{3}\right)$

$V=\frac{2000}{3}$

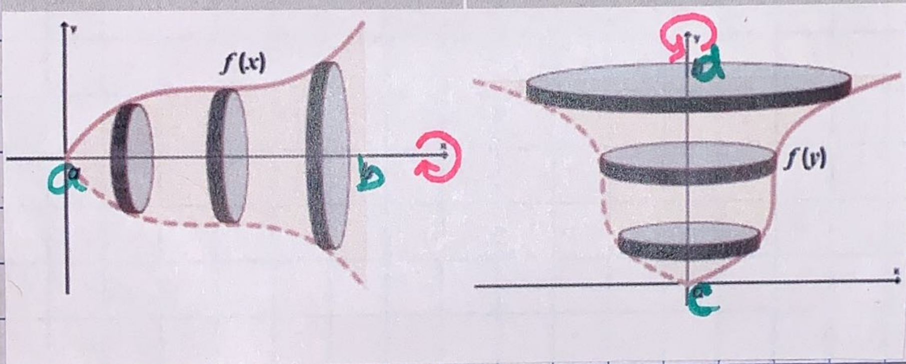
Volumes of Revolutions (7.4)

Disk method: To find the volume of a solid of revolution with the disk method, use one of the following:

Horizontal Axis of Revolution	Vertical Axis of Revolution
$Volume = V = \pi \int_a^b [R(x)]^2\,dx$	$Volume = V = \pi \int_c^d [R(y)]^2\,dy$

Area of disk is πr^2

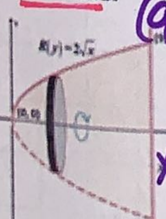
over x ↓ dx



over y ↓ dy

Examples:

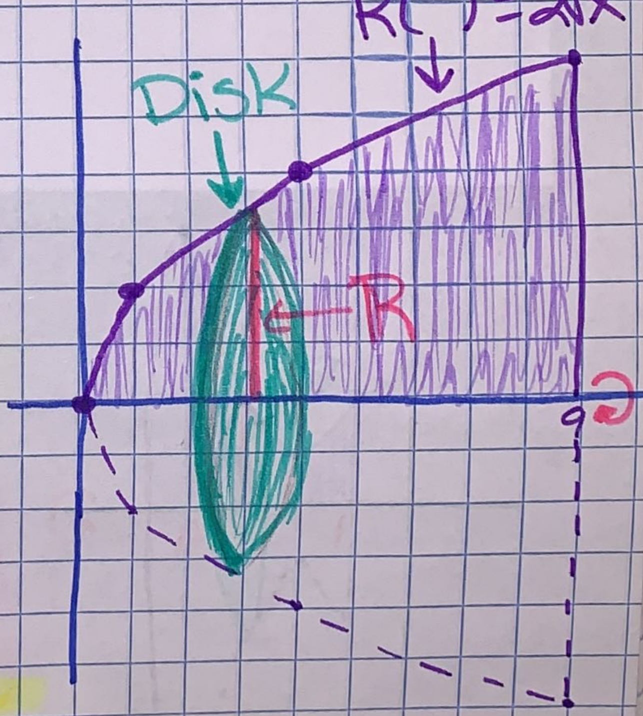
1. Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = 2\sqrt{x}, y = 0$ & $x = 9$ about the $x-axis$.



R = top - bottom

$R = 2\sqrt{x} - 0$

$R = 2\sqrt{x}$



$V = \pi \int_0^9 (2\sqrt{x})^2\,dx$

$V = \pi \int_0^9 4x\,dx$

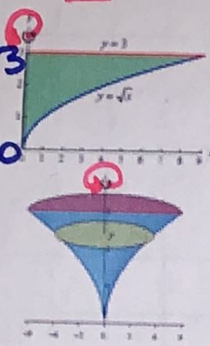
$V = \pi \left(2x^2 \Big|_0^9\right)$

$V = \pi(162 - 0)$

$V = 162\pi$

90

2. Determine the volume of the solid by rotating the region bounded by $y = \sqrt{x}$ and $y = 3$, about the $y-axis$.



R = right − left

Rewrite in terms of y

$y = \sqrt{x} \rightarrow x = y^2$

$R = y^2 - 0$ so $R = y^2$

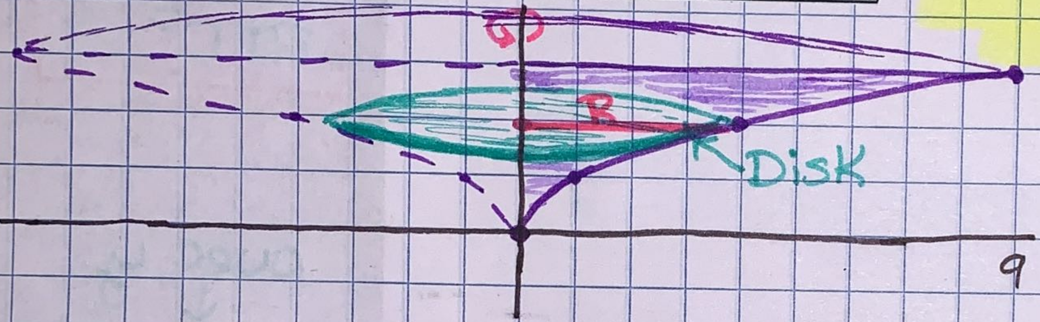
$V = \pi \int_0^3 (y^2)^2\,dy$

$V = \pi \int_0^3 y^4\,dy$

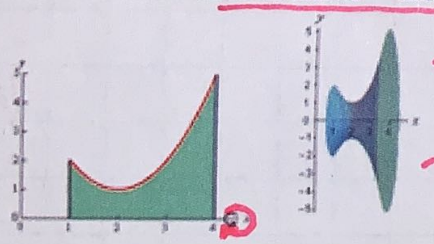
$V = \pi \left(\frac{y^5}{5} \Big|_0^3 \right)$

$V = \pi \left(\frac{243}{5} - \frac{0}{5} \right)$

$V = \frac{243\pi}{5}$



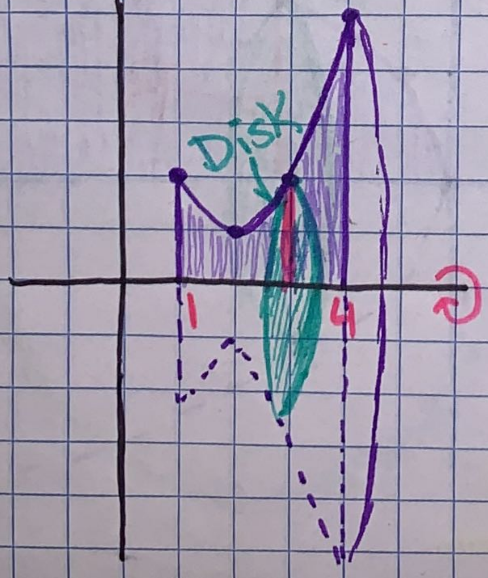
3. Determine the volume of the solid by rotating the region bounded by $y = x^2 - 4x + 5, x = 1, x = 4$, and the $x - axis$ about the $x - axis$.



dx

R = top − bottom

$R = x^2 - 4x + 5$



$V = \pi \int_1^4 (x^2 - 4x + 5)^2\,dx$

$V = \frac{78\pi}{5}$ or 15.6π

★ Calculator:

$\boxed{\text{math}}$ 9 → $\int_{\square}^{\square} (\square)\,d\square$

old calc: $\boxed{\text{math}}$ 9

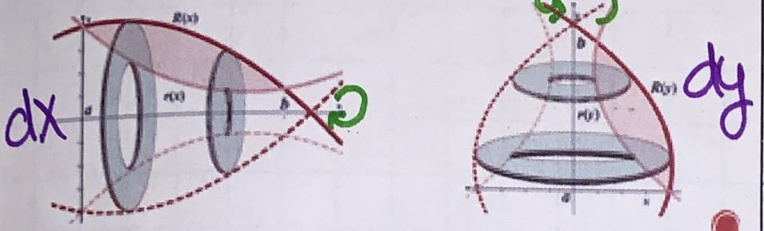
$\text{fnInt}(\underbrace{(x^2-4x+5)^2}_{\text{function}}, x, \text{lower bound}, \text{upper bound})$

Washer Method: use for solids of revolutions with holes

THE WASHER METHOD

Use the washer method for solids of revolution with holes.

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right)\, dx$$

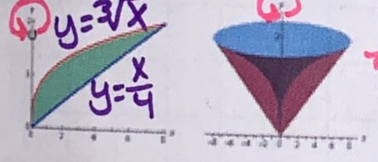


$R(x)$ = outer radius

$r(x)$ = inner radius

Examples:

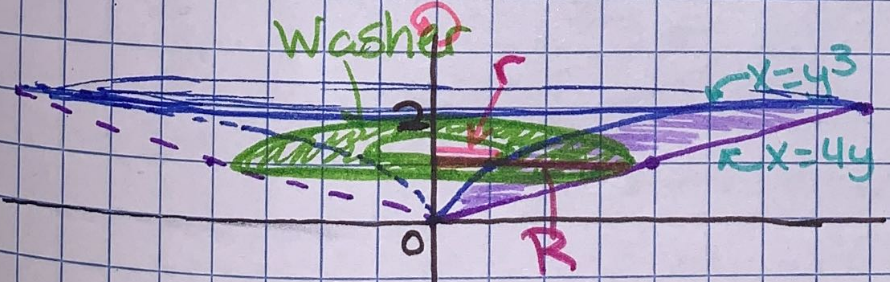
1. Determine the volume of the solid by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the $y-axis$. dy



*Rewrite in terms of y

$y = \sqrt[3]{x}$ so $x = y^3$

$y = \frac{x}{4}$ so $x = 4y$



$R(x) = 4y$

$r(x) = y^3$

$V = \pi \int_0^2 \left[(4y)^2 - (y^3)^2\right] dy$

$V = \pi \int_0^2 16y^2 - y^6\, dy$

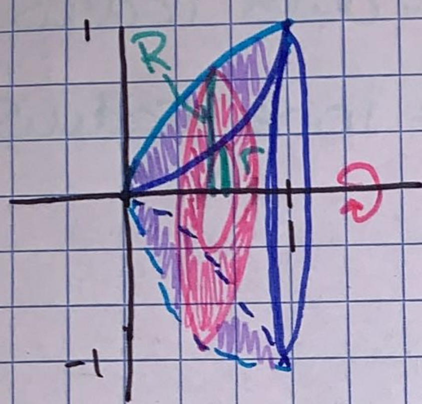
$V = \pi \left(\frac{16y^3}{3} - \frac{y^7}{7} \Big|_0^2\right)$

$V = \pi \left(\frac{128}{3} - \frac{128}{7} - 0 + 0\right)$

$V = \frac{512\pi}{21}$

Q2

2. Find the volume of the solid formed by revolving the region bounded by the graphs of $y=\sqrt{x}$ and $y=x^2$ about the x-axis.



$R(x)=\sqrt{x}$ $\quad$ $r(x)=x^2$

$V=\pi\int_0^1 \left[(\sqrt{x})^2-(x^2)^2\right]dx$

$V=\pi\int_0^1 x-x^4\,dx$

$V=\pi\left(\frac{x^2}{2}-\frac{x^5}{5}\Big|_0^1\right)$

$V=\pi\left(\frac{1}{2}-\frac{1}{5}-0+0\right)$

$V=\frac{3\pi}{10}$