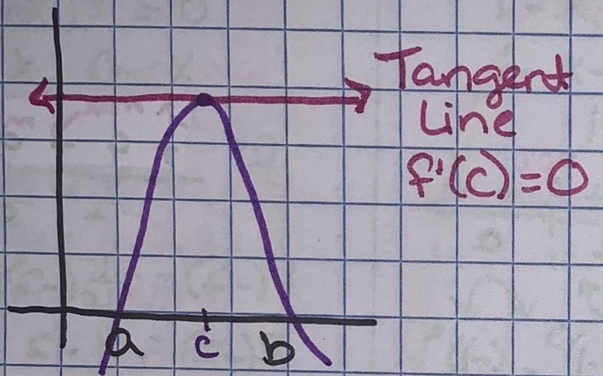
50 Mean Value Theorem (+ Rolle's Theorem)

Rolle's Theorem:

Suppose $f(x)$ is a function that satisfies all of the following:

1. $f(x)$ is continuous on the closed interval $[a,b]$
2. $f(x)$ is differentiable on the open interval (a,b)
3. $f(a) = f(b)$ (same y)

Then there is a number, c, such that $a < c < b$ + $f'(c) = 0$.
In other words, $f(x)$ has a critical pt. (max/min) in (a,b)



Basically, if a function is continuous + differentiable on an interval + the endpts have the same y-value, there is guaranteed to be at least one max/min within the interval (deriv=0)

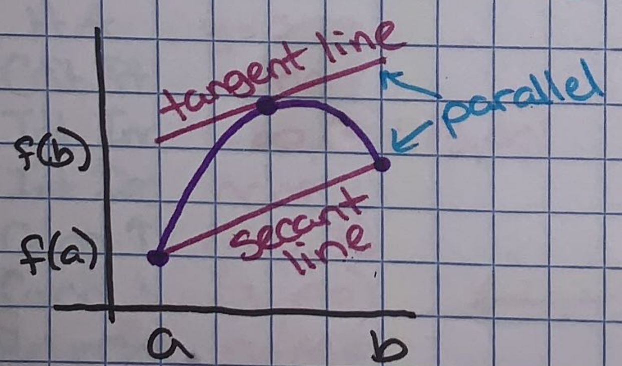
Mean Value Theorem:

Suppose $f(x)$ is a function that satisfies both of the following:

1. $f(x)$ is continuous on the closed interval $[a,b]$
2. $f(x)$ is differentiable on the open interval (a,b)

Then there is a number, c, such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Basically, if a function is continuous + differentiable on an interval, then somewhere in $[a,b]$ there is a tangent line that is parallel to the secant line connecting the endpts

Note: Rolle's Theorem is a special case of Mean Value Theorem

1. Let f be a function given by $f(x)=x^3-3x^2$. What are all the values of c that satisfy the conclusion of the Mean Value Theorem on the closed interval [0,3]?

need to say this

f(x) is continuous on [0,3]

f(x) is differentiable on (0,3)

Therefore, there exist a 'c' in (0,3) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$f(\underset{a}{0})=\underset{f(a)}{0}$ $f(\underset{b}{3})=\underset{f(b)}{0}$ $f'(c)=3c^2-6c$

$$3c^2-6c=\frac{0-0}{3-0}$$

$$3c^2-6c=0$$

$$3c(c-2)=0$$

$$c=0 \quad c=2$$

c=0 isn't in (0,3)

so only

$c=2$

2. Let f(x) = cos 2x. Find all values of c that satisfy the conclusion of the MVT for $0 \le x \le 2\pi$

f(x) is cont. on $[0,2\pi]$

f(x) is diff. on $(0,2\pi)$

Therefore exists in the set

$\therefore \exists c \in (0,2\pi)$ $f'(c)=\frac{f(b)-f(a)}{b-a}$

$f(0)=\cos(2\cdot 0)$

$f(0)=\cos 0=1$

$(0,1)$

$f(2\pi)=\cos(2\cdot 2\pi)$

$f(2\pi)=\cos 4\pi=1$

$(2\pi,1)$

$f'(x)=-\sin(2x)\cdot 2$

$f'(x)=-2\sin(2x)$

$$-2\sin(2x)=\frac{1-0}{2\pi-0}$$

$$-2\sin(2x)=0$$

$$\sin(2x)=0$$

$$2x=\sin^{-1}0$$

$2x=0$ + $2x=\pi$

$x=0$ $x=\frac{\pi}{2}$

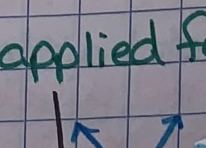
not in open interval

$c=\frac{\pi}{2}$

3. Determine whether MVT can be applied for $f(x)=|x-4|$ on [2,5]

f(x) is cont. on [2,5]

f(x) is not diff. on [2,5]



corner (not diff. at x=4)

MVT cannot be applied (or Rolle's)

Particle Motion

- $s(t)$ is the position of the particle moving along the x-axis
- $s'(t) = v(t)$: the 1st derivative is the velocity
- $s''(t) = a(t)$: the 2nd derivative is acceleration
- $v(t) = 0$ when the particle is at rest
- + velocity means the particle is moving right (or up)
- − velocity means the particle is moving left (or down)
- If $v(t)$ and $a(t)$ have the same signs, the particle is speeding up
- If $v(t)$ and $a(t)$ have different signs, the particle is slowing down
- Displacement is the change in position from beginning to end
- Total distance includes all of the distance traveled taking into consideration the particle can change directions.

$$\boxed{1.\ s(t) = t^3 - 6t^2 + 9t}$$

a) Find velocity at time t. $v(t) = 3t^2 - 12t + 9$

b) What is the velocity after 2 sec. $v(2) = 3(2)^2 - 12(2) + 9$

$v(2) = -3$ m/s

c) Find the acceleration as a function of time t.

$a(t) = 6t - 12$

d) Find the acceleration at t = 3 sec. $a(3) = 6(3) - 12$

$a(3) = 6$ m/s²

e) When is the particle at rest? $v(t) = 3t^2 - 12t + 9$

$0 = 3(t-3)(t-1)$

t = 3 sec t = 1 sec

f) When is the particle moving forward (right)? backward (left)?

$v(t)$ + 0 − 0 +
        1   3

moving forward: $(0,1) \cup (3,\infty)$

moving backward: $(1,3)$

g) What is the displacement on [0,5] sec?

$\underset{\text{beg}}{s(0)} = 0$ $\underset{\text{end}}{s(5)} = 20$ end − beg → 20 − 0 = 20m

h) Find the total distance traveled on [0,5] sec.

* It changed directions at t = 1, t = 3

$[0,1] \to s(1) - s(0) = |+4+0| = |+4| = 4$m

$[1,3] \to s(3) - s(1) = |0-4| = |-4| = 4$m 28m

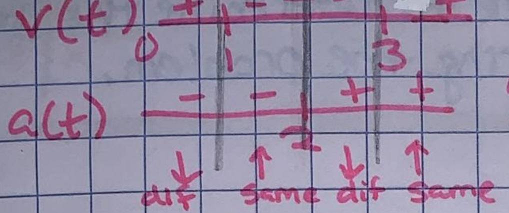
$[3,5] \to s(5) - s(3) = 20 - 0 = +20$m

i) Find the velocity when acceleration is $24 m/s^2$?

$a(t) = 6t - 12$
$24 = 6t - 12$
$t = 6$

$v(6) = 3(6)^2 - 12(6) + 9$
$v(6) = 45 m/s$

j) Find when the particle is speeding up + slowing down



$0 = 6t - 12$
$t = 2$

Speed up: $(1,2) \cup (3,\infty)$
Slow down: $(0,1) \cup (2,3)$

2. $s(t) = t^3 - 12t^2 + 45t$ on $[0,7]$

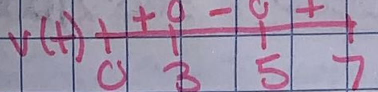
a) Velocity? Velocity at $t=2$?

$v(t) = 3t^2 - 24t + 45$
$v(2) = 9 m/s$

b) When is the particle at rest?

$0 = 3(t^2 - 8t + 15)$
$0 = 3(t-5)(t-3)$
$t = 5$ sec $t = 3$ sec

c) When is the particle moving right? moving left?



right: $(0,3) \cup (5,7)$
left: $(3,5)$

d) acceleration? acceleration at $t=1$?

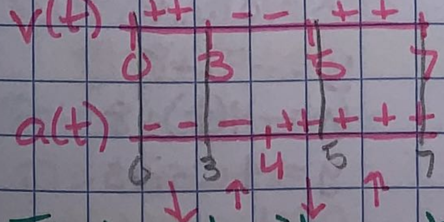
$a(t) = 6t - 24$
$a(1) = -18 m/s^2$

e) displacement? total distance traveled?

$s(0) = 0$
$s(7) = 70$
displacement = 70m

$[0,3]$ $s(0) = 0$ $s(3) = 54$ 54m
$[3,5]$ $s(3) = 54$ $s(5) = 50$ 4m
$[5,7]$ $s(5) = 50$ $s(7) = 70$ 20m
Total dist = 78m

f) speeding up? slowing down?



$6t - 24 = 0$
$t = 4$

Speed up: $(3,4) \cup (5,7)$
Slow down: $(0,3) \cup (4,5)$

g) Find velocity when acceleration is $0 m/s^2$.

$a(t) = 6t - 24$
$0 = 6t - 24$
$t = 4$ sec

$v(4) = 3(4)^2 - 24(4) + 45$
$v(4) = -3 m/s$

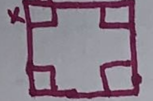
Optimization (max/min)

Strategies for Solving max/min problems

1. Draw + label a picture if relevant.
2. Translate the problem to an equation (with only 1 variable) that represents what you are trying to maximize or minimize.
3. Find the derivative & use an f' line to determine the max/min. Use that value to finish answering the problem.

EXAMPLE 1

From a thin piece of cardboard that is 6" x 6", square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?



$V = LWH$

$V = x(6-2x)(6-2x)$

$V = 36x - 24x^2 + 4x^3$

$L = 6-2x$

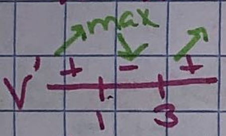
$W = 6-2x$

$H = x$

$V' = 36 - 48x + 12x^2$

$0 = 12(x^2 - 4x + 3)$

$0 = 12(x-3)(x-1)$



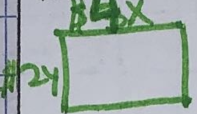
$x = 3 \quad x = 1 \leftarrow \text{max}$

L and W = 6 - 2(1) = 4"

4" x 4" x 1" $\quad V = 16\text{in}^3$

EXAMPLE 2

A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy fencing selling for \$4 a foot. While the remaining two sides will use standard fencing selling for \$2 a foot. What are the dimensions of the rectangular plot of greatest area that can be fenced at a cost of \$8000.



$P = 2L + 2W$

$8000 = 2(4x) + 2(2y)$

$8000 = 8x + 4y$

x = heavy

y = standard

$y = -2x + 2000$

$A = x \cdot y$

$A = x(-2x + 2000)$

$A = -2x^2 + 2000x$

$A' = -4x + 2000$

$0 = -4x + 2000$

$x = 500$

$y = -2(500) + 200$

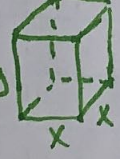
$y = -1000 + 200$

$y = 1000$

500' x 1000'

EXAMPLE 3

A soup company is constructing an open-top, square based, rectangular metal tank that will have a volume of 32 cubic feet. What dimensions yield the minimum surface area? What is the minimum surface area?



$V = LWH$

$32 = x \cdot x \cdot y$

$32 = x^2 y$

$y = \frac{32}{x^2}$

$SA = x^2 + 4xy$

$SA = x^2 + 4x\left(\frac{32}{x^2}\right)$

$SA = x^2 + 128x^{-1}$

$SA' = 2x - 128x^{-2}$

$0 = 2x - \frac{128}{x^2}$

$\frac{128}{x^2} = 2x$

$2x^3 = 128$

$x^3 = 64$

$x = 4'$

$y = \frac{32}{4^2}$

$y = 2'$

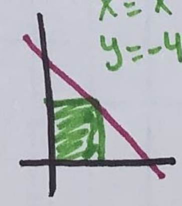
4' x 4' x 2'

$SA = (4)^2 + 4(4)(2)$

$SA = 48\text{ft}^2$

EXAMPLE 4

Find the rectangle of maximum area which is inscribed in the closed region bound by the x-axis and y-axis and the line y = -4x + 8.



$x = x$

$y = -4x + 8$

$A = L \cdot W$

$A = x(-4x + 8)$

$A = -4x^2 + 8x$

$A' = -8x + 8$

$0 = -8x + 8$

x = 1 unit

$y = -4x + 8$

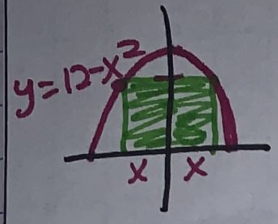
$y = -4(1) + 8$

y = 4 units

$A = (1)(4) = 4\text{ units}^2$

EXAMPLE 5

A rectangle has its base on the x-axis and its upper 2 vertices on the parabola $y = 12 - x^2$. What is the largest area that the rectangle can have and what are its dimensions?



$L = 2x$
$W = 12 - x^2$

$A = L \cdot W$
$A = 2x(12 - x^2)$
$A = 24x - 2x^3$
$A' = 24 - 6x^2$

$0 = 24 - 6x^2$
$6x^2 = 24$
$x^2 = 4$
$x = \pm 2$
$x = 2$ unit

$L = 2(2) = 4$
$W = 12 - (2)^2 = 8$
4u x 8u
$A = 32u^2$

EXAMPLE 6:

A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit must be $p = 1000 - x$. The manufacturer also determines that the total cost of producing x units is given by $C(x) = 3000 + 20x$.

a. Find the total revenue $R(x)$.
b. Find the total profit $P(x)$.
c. How many units must the company produce and sell in order to maximize profit?
d. What is the maximum profit?
e. What price per unit must be changed in order to make the maximum profit?

a) Revenue = # items sold · price/item
$R(x) = x(1000 - x)$
$R(x) = -x^2 + 1000x$

b) Profit = Revenue − Cost
$P(x) = (-x^2 + 1000x) - (3000 + 20x)$
$P(x) = -x^2 + 980x - 3000$

c) Units need to sell to maximize profit
$P'(x) = -2x + 980$
$0 = -2x + 980$
$x = 490$ units

d) max Profit
$P(490) = -(490)^2 + 980(490) - 3000$
$P(490) = 237,100$
\$237,100

e) Price per unit
$p = 1000 - x$
$p = 1000 - 490$
$p = \$510$

EXAMPLE 7:

A university is trying to determine what price to charge for football tickets. At a price of \$6 per ticket, it averages 70,000 people per game. For every increase of \$1, it loses 10,000 people from the average number. Every person at the game spends an average of \$1.50 on concessions. What price per ticket should be charged in order to maximize revenue? How many people will attend at that price?

Revenue = Ticket Rev + Concession Rev
$= \left(\begin{smallmatrix}\#\text{ of}\\ \text{people}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{price}\\ \text{per}\\ \text{ticket}\end{smallmatrix}\right) + \left(\begin{smallmatrix}\#\text{ of}\\ \text{people}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{concession}\\ \text{per person}\end{smallmatrix}\right)$

$R(x) = \underbrace{(70,000 - 10,000x)(6 + 1x)}^{\text{Ticket Rev}} +$
$\underbrace{(70,000 - 10,000x)(1.5)}_{\text{Concession Rev.}}$

$R(x) = 420,000 + 10,000x - 10,000x^2 +$
$105,000 - 15,000x$
$R(x) = -10,000x^2 - 5,000x + 525,000$

$R'(x) = -20,000x - 5,000$
$0 = -20,000x - 5,000$
$5000 = -20,000x$
$x = -.25$ or −\$0.25

price per ticket: $p = 6 + 1x$
$p = 6 + 1(-.25)$
$p = \$5.75$

people attending at new pric
$70,000 - 10,000x$
$70,000 - 10,000(-.25)$
72,500 people

Implicit Differentiation

- Explicit equations are solved for y $y=3x^4-2x+5$
- Implicit equations have y mixed throughout $2xy=x^4y-2x+5y$
- used to find derivatives when there are 2 variables + you can't easily solve for y

Steps:
- Differentiate x terms as usual (Power, Product, Trig etc)
- Differentiate y terms + ALWAYS put dy/dx behind it
- Collect dy/dx terms on left side + everything else on right
- Factor out dy/dx
- Solve for dy/dx

Find the derivative.

1. $x^2+y^2=1$

$2x+2y\,dy/dx=0$

$2y\,dy/dx=-2x$

$dy/dx=\frac{-2x}{2y}$

$\frac{dy}{dx}=\frac{-x}{y}$

2. $y^2+x^2=2x$

$2y\,dy/dx+2x=2$

$2y\,dy/dx=2-2x$

$dy/dx=\frac{2-2x}{2y}$

$\frac{dy}{dx}=\frac{1-x}{y}$

Product

3. $xy+y^2=1$

$x\cdot 1\,dy/dx+y\cdot 1+2y\,dy/dx=0$

$x\,dy/dx+y+2y\,dy/dx=0$

$x\,dy/dx+2y\,dy/dx=-y$

$dy/dx\,(x+2y)=-y$

$\frac{dy}{dx}=\frac{-y}{x+2y}$

Product

4. $x^3-xy+y^3=1$

$3x^2-(x\,dy/dx+y)+3y^2\frac{dy}{dx}=0$

$3x^2-x\frac{dy}{dx}-y+3y^2\frac{dy}{dx}=0$

$-x\frac{dy}{dx}+3y^2\frac{dy}{dx}=-3x^2+y$

$\frac{dy}{dx}(-x+3y^2)=-3x^2+y$

$\frac{dy}{dx}=\frac{-3x^2+y}{-x+3y^2}$

5. $x = \tan y$

$1 = \sec^2 y \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

chain rule

6. $x + \sin y^2 = xy$

$1 + \cos(y^2) \cdot 2y \frac{dy}{dx} = x \frac{dy}{dx} + y$

$2y \cos(y^2) \frac{dy}{dx} - x \frac{dy}{dx} = y - 1$

$\frac{dy}{dx} (2y \cos(y^2) - x) = y - 1$

$$\frac{dy}{dx} = \frac{y-1}{2y\cos(y^2) - x}$$

7. Find the tangent line of $x^2y^2 = 9$ at $(\overset{x}{-1}, \overset{y}{3})$

$x^2 y^2 = 9$

$x^2 \cdot 2y \, dy/dx + y^2 \cdot 2x = 0$

$2x^2 y \, dy/dx = -2xy^2$

$dy/dx = \frac{-2xy^2}{2x^2y} = \frac{-y}{x}$

dy/dx at $(\overset{x}{-1}, \overset{y}{3}) = \frac{-y}{x} = \frac{-3}{-1} = 3 \leftarrow m$

$$y - 3 = 3(x+1)$$

$y - y_1 = m(x-1)$

8. Find the derivative of $2xy + \pi \sin y = 2\pi$ at $(1, \pi/2)$

$2xy + \pi \sin y = 2\pi$

$2x \cdot 1 \, dy/dx + y \cdot 2 + \pi \cos y \, dy/dx = 0$

$2x \, dy/dx + 2y + \pi \cos y \, dy/dx = 0$

$2x \, dy/dx + \pi \cos y \, dy/dx = -2y$

$dy/dx \, (2x + \pi \cos y) = -2y$

$dy/dx = \frac{-2y}{2x + \pi \cos y}$

dy/dx at $(1, \pi/2) = \frac{-2(\pi/2)}{2(1) + \pi \cos \pi/2} = \frac{-\pi}{2 + \pi(0)} = \frac{-\pi}{2} = m$

Tangent Line: $y - \frac{\pi}{2} = -\frac{\pi}{2}(x-1)$

Related Rates

Related Rates Organization

1. Read the Problem
2. Answer these questions:
   - Know What rate are you given? (i.e. dV/dt)
   - Find What rate are you asked to find?
   - When most of the time, you are given a time when you want to find the rate. This info is used ONLY AFTER the derivative is taken. If it isn't given, you don't need it.
3. Write an equation that relates the Know rate with the Find rate.
4. Take a derivative. Since we are concerned about When things occur, we will be taking derivatives with respect to time (t) & we need to do implicit differentiation.
5. Substitute the Known rate + the When time.
6. Evaluate & label your answer.

Cubes, Circles, Spheres & Squares

EX 1:

Joe inflates a spherical balloon. Air is entering the balloon at a rate of $15\frac{cm^3}{sec}$. How fast is the radius changing when the radius is 10 cm?

K: $dV/dt = 15\ cm^3/sec$

F: dr/dt

W: $r = 10$ cm

$V = 4/3\,\pi r^3$

$dV/dt = 4\pi r^2\, dr/dt$

$15 = 4\pi(10)^2\, dr/dt$

$15 = 400\pi\, dr/dt$

$dr/dt = \frac{15}{400\pi}$

$dr/dt = \frac{3}{80\pi}$ cm/sec

EX 2:

A pebble is thrown into a pond forming ripples whose radius increases at the rate of 4 in/sec. How fast is the area of the ripple changing when the radius is one foot?

K: $dr/dt = 4$ in/sec

F: dA/dt

W: $r = 1\text{ft} = 12$ in

$A = \pi r^2$

$dA/dt = 2\pi r\, dr/dt$

$dA/dt = 2\pi(12)(4)$

$dA/dt = 96\pi\ in^2/sec$

EX 3:

The radius of a circle is increasing at the rate of 2 in/sec. At what rate is the area increasing when the circumference of the circle is 12π in.?

K: $\frac{dr}{dt}$ = 2 in/sec

F: dA/dt

W: C = 12π in

use circumference to find radius

$C = 2\pi r$

$12\pi = 2\pi r$

$r = 6$ in

$A = \pi r^2$

$dA/dt = 2\pi r \, dr/dt$

$dA/dt = 2\pi(6)(2)$

$dA/dt = 24\pi$ in²/sec

Ex 4:

A circular cotton doily with radius 22 cm is inadvertently thrown in the dryer and starts shrinking so that the radius is decreasing at a rate of 2 cm/min. At what rate is the area enclosed by the circle decreasing 5 minutes after the doily is thrown the dryer?

K: dr/dt = −2 cm/min

F: dA/dt

W: r = 22cm at 0 ~~sec~~ min

r = 12cm at 5 min

min	cm
0	22
1	20
2	18
3	16
4	14
5	12

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$= 2\pi(12)(-2)$

$\frac{dA}{dt} = -48\pi$ cm²/min

EX 5:

A piece of ice cut in the shape of a cube melts uniformly so that its volume decreases at 3 cm^3/sec. How fast is the surface area decreasing when the edge of the cube is 5 cm.?

K: dV/dt = −3 cm³/sec

F: dS/dt

W: e = 5 cm so V = 125cm³

$V = e^3$

$e = \sqrt[3]{V}$

↑ use to replace in SA formula

$V = (5)^3$

$V = 125\text{cm}^3$

$S = 6e^2$

$S = 6(\sqrt[3]{V})^2$

$S = 6V^{2/3}$

$\frac{dS}{dt} = 4V^{-1/3}\frac{dV}{dt}$

$\frac{dS}{dt} = 4(125)^{-1/3}(-3)$

$\frac{dS}{dt} = \frac{-12}{5}$ cm²/sec

EX 6:

Air is escaping from a spherical balloon at the rate of 2 cm^3 per minute. How fast is the surface area shrinking when the radius is 1 cm?

K: dV/dt = −2 cm³/min

F: dS/dt

W: r = 1 cm

use V to find dr/dt

$V = 4/3\pi r^3$

$dV/dt = 4\pi r^2 \frac{dr}{dt}$

$-2 = 4\pi(1)^2 dr/dt$

$dr/dt = \frac{-2}{4\pi}$

$\frac{dr}{dt} = \frac{-1}{2\pi}$

↑ plug into

$S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi(1)\left(\frac{-1}{2\pi}\right)$

$\frac{dS}{dt} = -4 \frac{\text{cm}^2}{\text{sec}}$

EX 7:

The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change in (a) the area (b) the perimeter

K: $\frac{dL}{dt} = -2$ cm/sec $\frac{dw}{dt} = 2$ cm/sec

F: dA/dt and dP/dt

W: $L = 12$cm $W = 5$cm

a) change in Area

$A = L \cdot W$

$\frac{dA}{dt} = L\frac{dw}{dt} + W\frac{dL}{dt}$

$\frac{dA}{dt} = 12(2) + 5(-2)$

$\frac{dA}{dt} = 14$ cm²/sec

b) change in Perimeter

$P = 2L + 2W$

$\frac{dP}{dt} = 2dL/dt + 2dW/dt$

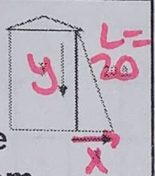
$\frac{dP}{dt} = 2(-2) + 2(2)$

$\frac{dP}{dt} = 0$ → Perimeter is constant

Related Rates - Triangles, Boats, Cars, & Ladders

Ex 1:

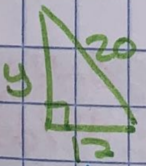
A 20 foot ladder is leaning against a house. The foot of the ladder begins to slide away from the house at a rate of 2 feet/second. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 12 feet from the house?



K: $dx/dt = 2$ ft/s

F: dy/dt

W: $x = 12$



① Find y

$x^2 + y^2 = L^2$

$12^2 + y^2 = 20^2$

$144 + y^2 = 400$

$y = 16$ft

② Find dy/dt

$x^2 + y^2 = L^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2L\frac{dL}{dt}$

$2(12)(2) + 2(16)\frac{dy}{dt} = 2(20)0$

ladder length is constant

$48 + 32\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-48}{32}$

$\frac{dy}{dt} = \frac{-3}{2}$ ft/sec

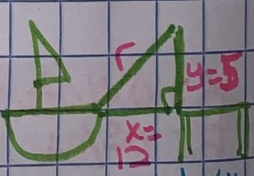
Ex 2:

A boat is pulled into a dock by a rope attached to it and passing through a pulley on the dock positioned 5 meters higher than the boat. If the rope is being pulled in at a rate of 2 m/sec, how fast is the boat approaching the dock when it is 12 meters away from the dock?

K: $dr/dt = -2$ m/sec

F: dx/dt

W: $x = 12$m



① Find r (rope)

$x^2 + y^2 = r^2$

$12^2 + 5^2 = r^2$

$r = 13$m

② Find dx/dt

$x^2 + y^2 = r^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

~~2(12)(2)+2(5)(~~

$2(12)(dx/dt) + 2(5)(0) = 2(13)(-2)$

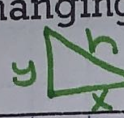
$24\,dx/dt + 0 = -52$

$\frac{dx}{dt} = \frac{-52}{24}$

$\frac{dx}{dt} = \frac{-13}{6}$ m/sec

Ex 3:

In a right triangle, leg x is increasing at the rate of 2 m/s while leg y is decreasing so that the area of the triangle is always equal to 6 m^2. How fast is the hypotenuse changing when $x = 3$ m?



$3^2+4^2=h^2$

$h=5$

K: $\frac{dx}{dt}=2$ m/s $\frac{dy}{dt}=$

F: dh/dt

W: $x=3$m

① Find y

$A=\frac{1}{2}xy$

$6=\frac{1}{2}(3)y$

$y=4$m

② Find $\frac{dy}{dt}$

$A=\frac{1}{2}xy$

$\frac{dA}{dt}=\frac{1}{2}x\frac{dy}{dt}+y\cdot\frac{1}{2}\frac{dx}{dt}$

constant area

$0=\frac{1}{2}(3)\frac{dy}{dt}+(4)(\frac{1}{2})(2)$

$0=\frac{3}{2}\frac{dy}{dt}+4$

$\frac{dy}{dt}=-\frac{8}{3}$ m/s

③ Find dh/dt

$x^2+y^2=h^2$

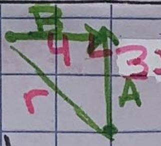
$2x\,dx/dt+2y\,dy/dt=2h\,dh/dt$

$2(3)(2)+2(4)(-8/3)=2(5)\,dh/dt$

$\frac{dh}{dt}=-14/15$ m/sec

Ex 4:

Cars A and B are approaching each other at an intersection. Car A is approaching north at 70 km/h & Car B is approaching east at 60 km/h. What rate are the cars approaching when car A is 3 km from the intersection & Car B is 4 km from the intersection?



K: $\frac{dA}{dt}=-70$ Km/hr

$\frac{dB}{dt}=-60$ Km/hr

F: dr/dt

W: A=3Km B=4Km

$A^2+B^2=r^2$

$2A\,dA/dt+2B\,dB/dt=2r\,dr/dt$

$2(3)(-70)+2(-60)(4)=2(5)\,dr/dt$

$-900=10\,dr/dt$

$\frac{dr}{dt}=-90$ Km/hr

Related Rates - Cones + Shadows

EX 1:

Assume that sand allowed to pour onto a level surface will from a pile in the shape of a cone, with height equal to the diameter of the base. If sand is poured at 2 cubic meters per second, how fast is the height of the pile increasing when the base is 8 meters in diameter?

K: $dV/dt=2m^3/sec$

F: dh/dt

W: d=8m so h=8m

$V=\frac{1}{3}\pi r^2h$

$V=\frac{1}{3}\pi(\frac{h}{2})^2h$

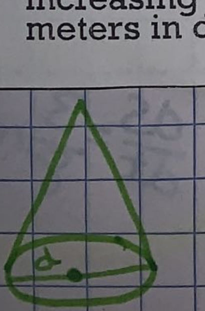
$V=\frac{1}{3}\pi\frac{h^2}{4}\cdot h$

$V=\frac{1}{12}\pi h^3$

$\frac{dV}{dt}=\frac{1}{4}\pi h^2\,dh/dt$

$2=\frac{1}{4}\pi(8)^2\,dh/dt$

$dh/dt=\frac{1}{8\pi}$ m/sec



$h=d$

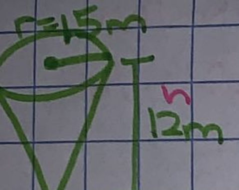
$h=2r$

$r=\frac{h}{2}$

★solve for r
+ Substitute
to have only
1 variable

EX 2:

A water tank has the shape of an inverted right-circular cone, with radius at the top 15 meters and depth 12 meters. Water is flowing into the tank at the rate of 2 cubic meters per minute. How fast is the depth of water in the tank increasing when the depth is 8 meters?



K: $dV/dt = 2m^3/min$

F: dh/dt

W: $h = 8m$

① Find r in terms of h

$\frac{r}{h} = \frac{15}{12}$

$15h = 12r$

$r = \frac{15}{12}h = \frac{5}{4}h$

② $V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{5}{4}h\right)^2 h$

$V = \frac{1}{3}\pi \frac{25}{16}h^2 \cdot h$

$V = \frac{25}{48}\pi h^3$

derive

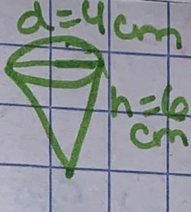
③ $dV/dt = \frac{25}{16}\pi h^2 dh/dt$

$2 = \frac{25}{16}\pi(8)^2 dh/dt$

$2 = 100\pi\, dh/dt \rightarrow \frac{dh}{dt} = \frac{1}{50\pi}$ m/min

EX 3:

A conical cup is 4 cm across and 6 cm deep. Water leaks out of the bottom at the rate of 2 cm^3/sec. How fast is the water level dropping when the height of the water is 3 cm?



K: $dV/dt = -2cm^3/sec$

F: dh/dt

W: $h = 3cm$

① $\frac{r}{h} = \frac{2}{6}$

$2h = 6r$

$r = \frac{1}{3}h$

② $V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$

$V = \frac{1}{27}\pi h^3$

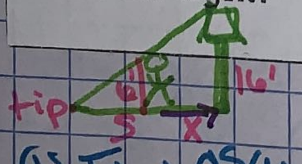
③ $dV/dt = \frac{1}{9}\pi h^2\, dh/dt$

$-2 = \frac{1}{9}\pi(3)^2\, dh/dt$

$\frac{dh}{dt} = \frac{-2}{\pi}$ cm/~~min~~ sec

EX 4:

A man 6 feet tall walks at the rate of 5 ft/sec toward a street light that is 16 ft above the ground. At what rate is the tip of the shadow moving? At what rate is the length of his shadow changing when he is 10 feet from the base of the light?



K: $dx/dt = -5ft/sec$

F: $dTip/dt$ & dS/dt

W: $x = 10ft$

① Find dS/dt

$\frac{6}{S} = \frac{16}{x+S}$

$16S = 6x + 6S$

$10S = 6x$

$10\, dS/dt = 6\, dx/dt$

$10\, dS/dt = 6(-5)$

$dS/dt = -3ft/sec$

② Find $dTip/dt$

$Tip = x + S$

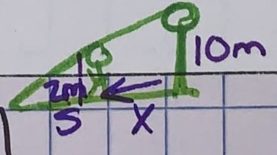
$dtip/dt = dx/dt + dS/dt$

$dtip/dt = -5 + -3$

$\frac{dtip}{dt} = -8 ft/sec$

EX 5:

A pickpocket walking away from a 10 meter tall lamppost is 2 meters tall. He walks at a rate of 1.5 m/sec. How fast is his shadow growing when he is 5 meter from the lamppost?



K: $dx/dt = 1.5 m/s$

F: ds/dt

W: $x = 5m$

$\frac{10}{x+s} = \frac{2}{s}$

$2x + 2s = 10s$

$2x = 8s$

$2\frac{dx}{dt} = 8\frac{ds}{dt}$

$2(1.5) = 8\frac{ds}{dt}$

$\frac{ds}{dt} = \frac{3}{8}$ m/sec