6.1 Indefinite Integration (Antiderivatives)

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Derivative	Antiderivative
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + C$

$\int \frac{1}{x}\,dx$ or $\int x^{-1}dx = \ln\lvert x\rvert + c$
$\int e^x dx = e^x + c$
$\int a^x dx = \frac{a^x}{\ln a} + c$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$
$\int \frac{1}{1+x^2}\,dx = \arctan x + c$
$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + c$

Always include +C!

C is the constant of integration

★ when trig answer starts with c, it's neg. (cos x, −cot x, −csc x)

Steps: ① Rewrite ② Integrate ③ Simplify

1. $\int \frac{1}{x^3}dx$
$\int x^{-3}dx$
$\frac{x^{-3+1}}{-3+1} + C$
$\frac{x^{-2}}{-2} + C$
$\frac{-1}{2x^2} + C$

2. $\int \sqrt{x}\,dx$
$\int x^{1/2}dx$
$\frac{x^{1/2+1}}{1/2+1} + C$
$\frac{x^{3/2}}{3/2} + C$
$\frac{2x^{3/2}}{3} + C$

3. $\int \frac{1}{x}dx$ ★ memorize! Can't work it out
$\ln\lvert x\rvert + C$

4. $\int \frac{13}{17}e^x$
$\frac{13}{17}\int e^x dx$
$\frac{13}{17}e^x + C$

5. $\int 2\pi \sin x\,dx$
$2\pi\int \sin x\,dx$
$2\pi(-\cos x) + C$
$-2\pi\cos x + C$

6. $\int \frac{1}{x\sqrt{x}}dx$
$\int x^{-3/2}dx$
$\frac{x^{-3/2+1}}{-3/2+1} + C$
$\frac{x^{-1/2}}{-1/2} + C$
$\frac{-2}{x^{1/2}} + C$ or $\frac{-2}{\sqrt{x}} + C$

7. $\int 3^x dx$ ★ memorize
$\frac{3^x}{\ln 3} + C$

8. $\int \frac{1}{2x^3}dx$
$\frac{1}{2}\int x^{-3}dx$
$\frac{1}{2}\cdot\frac{x^{-3+1}}{-3+1} + C$
$\frac{x^{-2}}{-4} + C$
$\frac{-1}{4x^2} + C$

9. $\int \frac{1}{(2x)^3}dx$
$\int \frac{1}{8x^3}dx$
$\frac{1}{8}\int x^{-3}dx$
$\frac{1}{8}\cdot\frac{x^{-2}}{-2} + C = \frac{-1}{16x^2} + C$

10. $\int \frac{1}{\sqrt{1-x^2}}dx$
$\sin^{-1}x + C$
need to recognize as a trig inverse derivative

11. $\int (x+3)(3x-2)dx$
$\int 3x^2+7x-6\,dx$
$\frac{3x^3}{3} + \frac{7x^2}{2} - 6x + C$
$x^3 + \frac{7}{2}x^2 - 6x + C$

Trig Id
12. $\int \frac{\sin x}{\cos^2 x}dx = \int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}dx$
$\int \tan x\cdot\sec x\,dx$
$\sec x + C$

13. $\int \frac{x^2+x+1}{\sqrt{x}}dx$ Rewrite 1st
$\int x^{3/2} + x^{1/2} + x^{-1/2}$
$\frac{x^{5/2}}{5/2} + \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$
$\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C$

6.2 Riemann Sums

What is an integral? The AREA under the curve!

Riemann Sums - approximates an integral (or area under curve)

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\,\Delta x$$

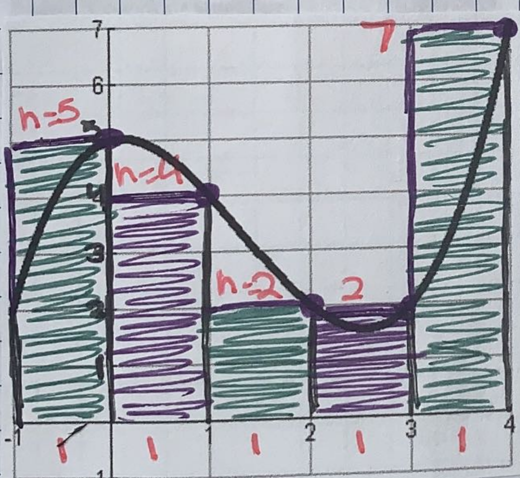
Right Riemman Sums (RSS): draw from right

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(a_i)\,\Delta x = \frac{b-a}{n}\left[f(x_1)+f(x_2)+\dots+f(x_n)\right]$$

Left Riemman Sums (LSS): draw from left

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(a_{i-1})\,\Delta x = \frac{b-a}{n}\left[f(x_0)+f(x_1)+\dots+f(x_n)\right]$$

1. Approximate the integral using the Right Endpt. method (5 subintervals)



Right R.S. approximation:

- Each rectangle has a base of 1 or $\frac{b-a}{n}$
- $A = b \cdot h$

$\approx 1(5) + 1(4) + 1(2) + 1(2) + 1(7)$

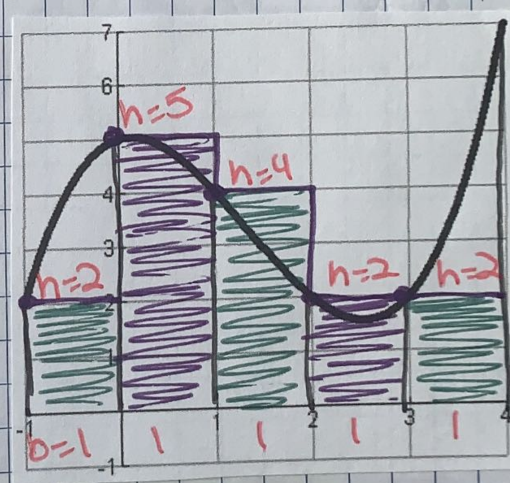
or

$\approx 1(5+4+2+2+7)$

$A \approx 20$ units^2

This appears to be an overapproximation.

2. Approximate the integral using the Left Endpoint method (5 subintervals)



Left R.S. approximation:

$A \approx 1(2+5+4+2+2)$

$A \approx 15$ units^2

Appears to be an underestimation

☆ The more subintervals you have, the more accurate it will be.

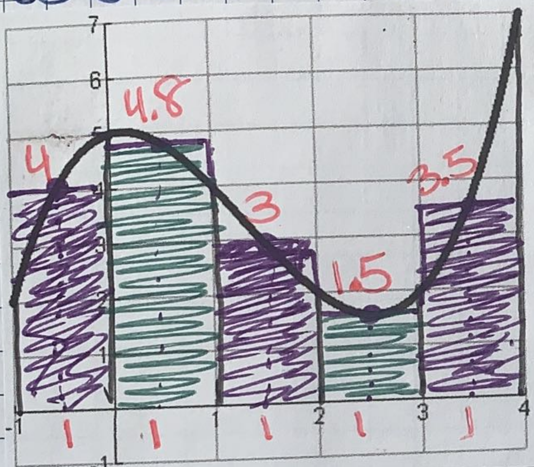
Midpoint Riemann Sums (MRS): draw from midpt of interval

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{a_{i-1}+a_i}{2}\right)\Delta x = \frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+\dots f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

Trapezoid Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n}\left(\frac{f(a_{i-1})+f(a_i)}{2}\right)\Delta x = \frac{1}{2}\cdot\frac{b-a}{n}\left[f(x_0)+2f(x_1)+\dots+2f(x_{n-1})+f(x_n)\right]$$

3. Approximate the integral using the midpt method (5 subintervals)

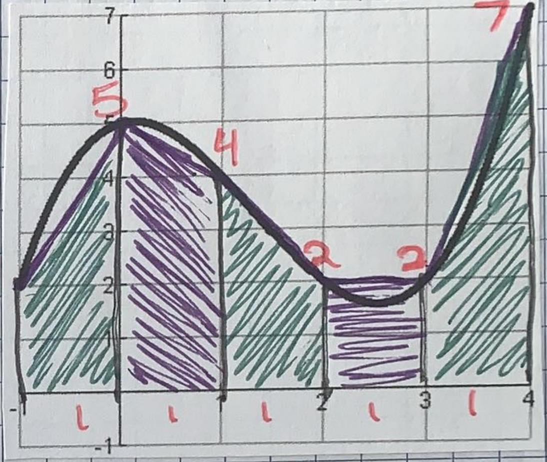


midpt. RS Approximation:

$A \approx 1(4+4.8+3+1.5+3.5)$

$A \approx 16.8$ units^2

4. Approximate the integral using the trapezoid method (5 subintervals)



Trapezoid RS Approx:

Each trapezoid has a base of 1

$A=\frac{1}{2}h(b_1+b_2)$

$A \approx \frac{1}{2}(1)\left[(2+5)+(5+4)+(4+2)+(2+2)+(2+7)\right]$

or $A \approx \frac{1}{2}(1)\left[2+2(5)+2(4)+2(2)+2(2)+7\right]$

$A \approx 17.5$ units^2

Circumscribed Method - Highest pt in interval used to create rectangles

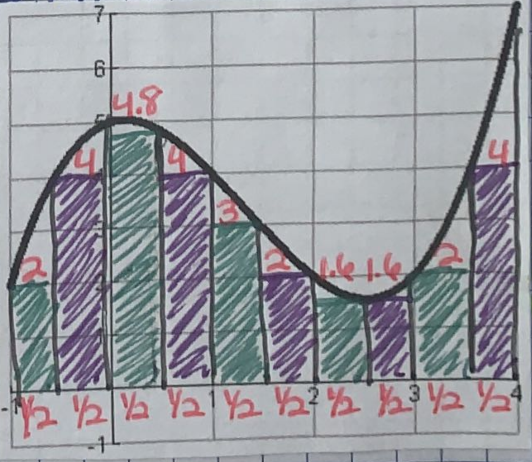
Inscribed Method - Lowest pt in interval used to create rectangles

5. Approx. Circumscribed (10 intervals)

$A=\frac{1}{2}(4+5+5+4.8+4+3+2+2+4+7)$

$A \approx 20.4$ units^2

6. Approx. Inscribed (10 intervals)



$A=\frac{1}{2}(2+4+4.8+4+3+2+1.6+1.6+2+4)$

$A \approx 14.5$ units^2

6.3 Riemann Sum to an Integral

$\Delta x = \frac{b-a}{n}$ (tells you base of ea. subinterval)

$x_k = a + k\Delta x$ (tells you length of base for total rectangles)

Rule: $\lim_{n\to\infty} \sum_{i=1}^{n} f(a + k\Delta x)\Delta x = \int_a^b f(x)\,dx$

1. State the integral the sum is approx. (RRS approx.)

$\frac{1}{4}\left[\sin\left(\frac{9}{4}\right) + \sin\left(\frac{10}{4}\right) + \sin\left(\frac{11}{4}\right) + \sin(3)\right]$

base of rectangles; don't include furthest left pt on RRS; furthest to right

$\Delta x = 1/4$

RRS

4 subintervals

end at 3 which is $\frac{12}{4}$ & begin at $2\left(\frac{8}{4}\right)$

$\int_2^3 \sin x\,dx$

2. RRS Approximation

$\frac{1}{8}\left[\frac{1}{(1\frac{1}{8})^2} + \frac{1}{(1\frac{1}{4})^2} + \frac{1}{(1\frac{3}{8})^2} + \frac{1}{(1\frac{1}{2})^2} + \frac{1}{(1\frac{5}{8})^2} + \frac{1}{(1\frac{3}{4})^2} + \frac{1}{(1\frac{7}{8})^2} + \frac{1}{(2^2)}\right]$

$\Delta x = 1/8$

RRS so 2 is furthest pt

8 subintervals

$\int_1^2 \frac{1}{x^2}\,dx$

3. LRS Approximation

$\frac{1}{4}\left[\frac{1}{2(1)} + \frac{1}{2(1.25)} + \frac{1}{2(1.5)} + \frac{1}{2(1.75)}\right]$

$\Delta x = 1/4$

LRS so 1 is the a but we don't know b

4 subintervals

$\int_1^2 \frac{1}{2x}\,dx$

4. Find integral represented by

$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\frac{j}{n}\right)^4$

$\lim_{n\to\infty} \sum_{i=1}^{n} f(a + j\Delta x)\Delta x$

$a + j\Delta x = 0 + \frac{j}{n}$

$a = 0,\ \Delta x = \frac{1}{n},\ f(x) = x^4$

$\int_a^b f(x)\,dx = \int_0^b x^4\,dx$

$\Delta x = \frac{b-a}{n} = \frac{b-0}{n} = \frac{1}{n}$

$b = 1$ $\int_0^1 x^4\,dx$

Find integral represented by

5. $\lim_{n\to\infty} \frac{3}{n} \sum_{j=1}^{n} \left(2 + \frac{3j}{n}\right)^4$

$\lim_{n\to\infty} \sum_{i=1}^{n} f(a + j\Delta x)\Delta x$

$a + j\Delta x = 2 + \frac{3j}{n}$ so

$a = 2,\ \Delta x = \frac{3}{n},\ f(x) = x^4$

$\int_a^b f(x)\,dx = \int_2^b x^4\,dx$

now Find b → $\Delta x = \frac{b-a}{n} = \frac{b-2}{n} = \frac{3}{n}$ so $b - 2 = 3$, $b = 3$

$\int_2^5 x^4\,dx$

6. $\lim_{n\to\infty} \frac{5}{n} \sum_{j=0}^{n-1} e^{-2+\frac{5j}{n}}$

$\lim_{n\to\infty} \sum_{i=1}^{n} f(a + j\Delta x)\Delta x$

$a + j\Delta x = -2 + 5j/n$

so $a = -2,\ \Delta x = \frac{5}{n}$

& $f(x) = e^x$

$\int_a^b f(x)\,dx = \int_{-2}^b e^x\,dx$

$\Delta x = \frac{b-a}{n} = \frac{b-(-2)}{n} = \frac{5}{n}$

$b + 2 = 5$ so $b = 3$

$\int_{-2}^3 e^x\,dx$

6.4 The Fundamental Theorem of Calculus

Steps for Evaluating Definite Integrals

To find the area under the graph of a nonnegative, continuous function f over the interval [a,b]

1. Find any derivative F(x) of f(x).
2. Evaluate F(x) using b and a, + compute F(b) − F(a). The result is the area under the graph over the interval [a,b]

1. Find the area under the graph of $y = x^2 + 1$ over the interval [-1,2].

$\int_{-1}^{2} x^2 + x \, dx = \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2}$

$= \left(\frac{2^3}{3} + \frac{2^2}{2}\right) - \left(\frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right)$

$= \frac{8}{3} + 2 + \frac{1}{3} - \frac{1}{2} = \frac{9}{2}$

FTC Part 2
Definite Integral Definition

Let f be any continuous fxn over [a,b] + F be any antiderivative of f. Then the definite integral of f from a to b is

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

2. $\int_0^3 e^x \, dx$

$e^x \Big|_0^3 = e^3 - e^0$

$= e^3 - 1$

3. $\int_1^e \left(1 + 2x - \frac{1}{x}\right)$

$= x + x^2 - \ln x \Big|_1^e$

$= (e + e^2 - \ln e) - (1 + 1 - \ln 1)$

$= e + e^2 - 3$

4. $\int_2^4 (2x^3 - 3x)\,dx$

$= \frac{x^4}{2} - \frac{3x^2}{2} \Big|_2^4$

$= (128 - 24) - (8 - 6)$

$= 102$

FTC Part 1: If f is cont. on [a,b], then the function F defined by $F(x) = \int_a^x f(t)\,dt$ for $a \le x \le b$ is an antiderivative of f that is $F'(x) = f(x)$ for $a \le x \le b$ or $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ for x b/n a + b.

5. $g(x) = \int_1^x t^2 \, dx = x^2$

$g(x) = \frac{t^3}{3}\Big|_1^x$ $\quad g'(x) = x^2$

$g(x) = \frac{x^3}{3} - \frac{1}{3}$ $\quad$ deriv. of the integral

6. FTC Shortcut:

$\frac{d}{dx} \int_{\pi}^{x} \frac{1}{1+t^4}\,dx$

$= \frac{1}{1+x^4}$

7. $\frac{d}{dx} \int_x^2 \cos(t^2)\,dt$ ★

$-1 \frac{d}{dx} \int_x^2 \cos(t^2)\,dt$ $\quad$ negate when x is on bottom

$= -\cos(x^2)$

Next level FTC:

8. $g(x) = \int_3^{x^2} \sin t \, dt$

$g(x) = -\cos t \Big|_3^{x^2}$

$-\cos x^2 + \cos 3$

$g'(x) = \sin(x^2) \cdot 2x + 0 = 2x \cdot \sin(x^2)$

9. $\frac{d}{dx} \int_3^{x^2} \sin^4 t \, dt$

$\sin^4(x^2) \cdot 2x$

$2x \cdot \sin^4(x^2)$

10. $\frac{d}{dx} \int_1^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds$

$\frac{(\sqrt{x})^2}{(\sqrt{x})^2 + 1} \cdot \frac{1}{2\sqrt{x}}$

$\frac{x}{2\sqrt{x}\,(x+1)}$

Top level FTC:

11. $\frac{d}{dx} \int_{2x}^{3x} \frac{u-1}{u+1}\,du$

$3\left(\frac{3x-1}{3x+1}\right) - 2\left(\frac{2x-1}{2x+1}\right)$

12. $\frac{d}{dx} \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^2}}\,dt$

$2x\left(\frac{1}{\sqrt{2+(x^2)^2}}\right) - \left(\frac{1}{\sqrt{2+\tan^2 x}}\right)\sec^2 x$

$\frac{2x}{\sqrt{2+x^4}} - \frac{\sec^2 x}{\sqrt{2+\tan^2 x}}$

$\int$ velocity $\frac{miles}{hr}$ = miles $\int$ acceleration $\frac{m}{sec^2}$ = m/sec

1. At 7am water begins leaking from a tank at a rate of leaking = 2t + .25t gal/hr. (t is the # of hrs after 7am). How much water is lost between 9am + 11am?

$\int_2^4 2t + .25t = \int_2^4 \frac{5}{4}t\,dt = \frac{5t^2}{8}\Big|_2^4 = \frac{5(4)^2}{8} - \frac{5(2)^2}{8} = 13.5$

13.5 gallons of water lost between 9am + 11am

EXAMPLE 2

The table given below represents the velocity of a particle at given values of t, where t is measured in minutes.

3 subintervals 10 10 10

t minutes	0	5	10	15	20	25	30
$v(t)$ ft/minute	0	1.6	2.7	3.1	2.4	1.6	0

a. Approximate the value of $\int_0^{30} v(t)\,dt$ using a midpoint Riemann Sum. Using correct units of measure, explain what this value represents.

b. What is the value of $\int_5^{25} a(t)\,dt$, and using correct units, explain what this value represents.

a. $\int_0^{30} v(t)\,dt \approx 10[1.6 + 3.1 + 1.6]$ b h h h

≈ 63 ft

This represents the amt of feet traveled by the particle from 0-30 minutes

b. $\int_5^{25} a(t)\,dt = v(t)\Big|_5^{25} = v(25) - v(5)$

$= 1.6 - 1.6 = 0$ ft/min

represents Δv from 5 to 25 min.

EXAMPLE 3

A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. During the first 5 days of a 60-day period, 3 millimeters of rainfall had been collected. The height of water in the can is modeled by the function, S, where $S(t)$ is measured in millimeters and t is measured in days for $5 \le t \le 60$. The rate at which the height of the water is rising is given by the function $S'(t) = 2\sin(0.03t) + 1.5$.

a. Find the value of $\int_{10}^{15} S'(t)\,dt$. Using correct units, explain the meaning of this value in the context of this problem.

b. At the end of the 60-day period, what is the volume of water that had accumulated in the can? Show your work.

a. $\int_{10}^{15} 2\sin(0.03t) + 1.5\,dt$ Calc! CTRL+

≈ 11.159 mm

This represents the Δ water level (in mm) from 10-15 days

b) $\int_5^{60} 2\sin(0.03t) + 1.5\,dt$ calc.

$= 163.565$ mm from day 5 to 60

163.565 + 3mm to start =

$= 168.565$ mm accumulated in the 60 day period

EXAMPLE 4

The rate at which people enter an auditorium for a concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. VIP tickets were sold to 100 people who are already in the auditorium when the doors open at $t = 0$ for general admission ticket holders to enter. The doors close and the concert begins at $t = 2$.

If all of the VIP ticket holders stayed for the start of the concert, how many people are in the auditorium when the concert begins?

$\int_0^2 (1380t^2 - 675t^3)\,dt$ calculator

= 980 ppl from 0 to 2 hrs.

980 ppl + 100 VIP before start

1080 people in the auditorium

Displacement vs Total Distance

Displacement: how far from home (+/-) from where you started

$\int_{t_1}^{t_2} v(t)\,dt = \int_{t_1}^{t_2} \frac{\text{meas of length}}{\text{meas of time}}\,dt$ = meas of length

Total Distance: how far traveled

$\int_{t_1}^{t_2} |v(t)|\,dt = \int_{t_1}^{t_2} \left|\frac{\text{meas. length}}{\text{meas time}}\right| dt$ = meas of length

5. Assume a particle moves along a straight line with given velocity. Find the total displacement + total distance over the time interval.

$f(x) = x^2 - 2x - 4$ [-5, 5]

Displacement: $\int_{-5}^{5} x^2 - 2x - 4\,dx =$

$\frac{x^3}{3} - x^2 - 4x\Big|_{-5}^{5} = \left(\frac{125}{3} - 25 - 20\right) - \left(\frac{-125}{3} - 25 + 20\right)$

$\frac{130}{3}$ units

Total Dist.: $\int_{-5}^{5} |x^2 - 2x - 4|\,dx$ *use calc!

= 74.148